ARPM Certificate

Level 1

Example with solutions

Solution 1 True or false? Justify your answers
Note: 1 point is assigned to correct True/False; 3 points are assigned to a correct motivation.

1. *[4 points]* The lognormal distribution is stable.

   False. Since the lognormal distribution has finite variance, the sum of i.i.d. lognormal converges to a normal distribution, which is inconsistent with the definition of stability.

2. *[4 points]* The sample mean - sample covariance ellipsoid is the minimum volume ellipsoid that includes all the data points.

   False. The average Mahalanobis distance of the data from the center of the ellipsoid is 1, meaning that some observations are inside and some outside of the ellipsoid.

3. *[4 points]* In a GARCH(1,1) process, the next-step volatility conditional on the current information is deterministic.

   True. In a GARCH(1,1) process, the next-step volatility conditional on the current information is
   $$\Sigma_{t+1}|\boldsymbol{i}_t = \sqrt{c + b\sigma_t^2 + a\left(x_t - x_{t-1} - \mu\right)^2} = \sigma_{t+1},$$
   where we use the convention that upper case variables are random and lower case variables are known/deterministic.

4. *[4 points]* The Ornstein-Uhlenbeck process is the continuous time generalization of the AR(1) process with normal shocks.

   True. The dynamics of a OU process integrated over any finite interval is an AR(1) process with normal shocks.

5. *[4 points]* The r-squared $\mathcal{R}^2\{Y\|X\} \equiv 1 - \frac{\mathbb{E}\{(X-Y)^2\}}{\mathbb{V}\{X\}}$ is a divergence.

   False. The r-squared $\mathcal{R}^2\{Y\|X\}$ is not a divergence: it can become negative, and, if $X = Y$, it is equal to 1.

6. *[4 points]* If we apply principal component analysis to a highly correlated market, the first three principal components can be interpreted as level, slope and curvature, due to the shape of the eigenvectors.

   False. The eigenvectors take the shape of trigonometric waves if the covariance matrix has a Toeplitz structure, not if the correlation is high.

7. *[4 points]* Binary classification problems are instances of supervised learning where the input is a Bernoulli random variable.

   False. The output is a Bernoulli random variable, the input(s) may or may not be categorical.

8. *[4 points]* The ROC curve lies on the unit square and always starts from the point $(1, 1)$ and ends in the point $(0, 0)$.

   True. The ROC curve is defined as

   $$roc(\bar{s}) \equiv (fpr(\bar{s}), tpr(\bar{s})), \qquad \bar{s} \in (-\infty, +\infty). \tag{1}$$

   where the false and true positive rates are probabilities (between 0 and 1), hence the curve lies in the unit square.
   The curve starts at $(1, 1)$ because for $\bar{s} \to -\infty$ everything is classified as positive, so both the false and the true positive rates are equal to 1.
   The curve ends at $(0, 0)$ because for $\bar{s} \to +\infty$ everything is classified as negative, so both the false and the true positive rates are equal to 0.

9. *[4 points]* Feature engineering is the process of transforming input variables to achieve a model with better out of sample performance (variance reduction).

   False. Feature engineering is the process of transforming input variables to achieve a model with better in-sample fit (bias reduction).

10. *[4 points]* Lasso is a regularization technique that can be used to perform factor selection in a regression linear factor model.

    True. By suitably calibrating the penalty parameter, we can select $\bar{k}$ factors out of a larger pool, as the lasso optimization shrinks the loadings of the remaining factors to 0. See Lasso regression.

Solution 2 Generating scenarios from a copula-marginal distribution

[20 points] Using pseudo-code, write the steps to generate scenarios representing a random variable $\boldsymbol{X} \equiv (X_1, X_2)'$ with Cauchy copula $Cauchy(\boldsymbol{\varrho}^2)$ and the following marginal distributions:
- $X_1 \sim Bernoulli(0.3)$
- $X_2 \sim N(1, 1)$.

You can assume that the following tools are available:
- a univariate uniform random number generator;
- a multivariate standard normal random number generator;
- cumulative distribution functions and quantile functions of the relevant univariate distributions;
- function that computes the Riccati root of a symmetric and positive semidefinite matrix.

1. Generate scenarios for the copula

```
# joint Cauchy scenarios via uniform-radial representation
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$\boldsymbol{\varrho} \leftarrow$`Riccati`$(\boldsymbol{\varrho}^2)$

$\{\tilde{u}^{(j)}\}_{j=1}^{\bar{\jmath}} \overset{MC}{\Leftarrow} Unif([0,1])$

$\{r^{(j)}\}_{j=1}^{\bar{\jmath}} \leftarrow \{q_R(\tilde{u}^{(j)})\}_{j=1}^{\bar{\jmath}}$ where $q_R = F_R^{-1}(x)$, and $F_R(x) = F_{3,1}^{F}(r^2/3)$ (see here)

$\{\boldsymbol{n}^{(j)}\}_{j=1}^{\bar{\jmath}} \overset{MC}{\Leftarrow} N(0, \mathbb{I}_2)$

$\{\boldsymbol{y}^{(j)}\}_{j=1}^{\bar{\jmath}} \leftarrow \{\boldsymbol{n}^{(j)} / \left\|\boldsymbol{n}^{(j)}\right\|\}_{j=1}^{\bar{\jmath}}$

$\{\tilde{\boldsymbol{x}}^{(j)}\}_{j=1}^{\bar{\jmath}} \leftarrow \{r^{(j)} \boldsymbol{\varrho} \boldsymbol{y}^{(j)}\}_{j=1}^{\bar{\jmath}}$

```
# copula scenarios
```
$\{\boldsymbol{u}^{(j)}\}_{j=1}^{\bar{\jmath}} \leftarrow \{(F_{0,1}^{Cauchy}(\tilde{x}_1^{(j)}), F_{0,1}^{Cauchy}(\tilde{x}_2^{(j)}))'\}_{j=1}^{\bar{\jmath}}$

2. Map the marginal distributions into the desired distributions

$\{x_1^{(j)}\}_{j=1}^{\bar{\jmath}} \leftarrow \{1_{\{u_1^{(j)} \geq 0.7\}}\}_{j=1}^{\bar{\jmath}}$

$\{x_2^{(j)}\}_{j=1}^{\bar{\jmath}} \leftarrow \{q_{1,1}^{N}(u_2^{(j)})\}_{j=1}^{\bar{\jmath}}$

Solution 3 Regression linear factor model

Consider a regression linear factor model for an $(\bar{n} = 2)$-dimensional target variable $\boldsymbol{X}$ with $\bar{k} = 1$ factor Z

$$\boldsymbol{X} = \boldsymbol{\alpha} + \boldsymbol{\beta} Z + \mathring{\boldsymbol{\varepsilon}}. \tag{2}$$

Assume that the target variables and factor have the following joint Student t distribution

$$\begin{pmatrix} X_1 \\ X_2 \\ Z \end{pmatrix} \sim t(\underbrace{\begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}}_{\boldsymbol{\mu}}, \underbrace{\begin{pmatrix} 10 & 0 & 6 \\ 0 & 6.75 & 6.5 \\ 6 & 6.5 & 20 \end{pmatrix}}_{\boldsymbol{\sigma}^2}, \underbrace{4}_{\nu}). \tag{3}$$

1. *[10 points]* Compute the loadings $\boldsymbol{\beta} \equiv (\beta_1, \beta_2)$ that maximize the multivariate r-squared and the shift parameters $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2)'$ such that the residuals have zero expectation.

   The expectation and covariance of the joint distribution of $\boldsymbol{X}$ and Z are

$$\mathbb{E}\{\begin{pmatrix} X_1 \\ X_2 \\ Z \end{pmatrix}\} = \boldsymbol{\mu} \equiv \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}, \quad \mathbb{C}v\{\begin{pmatrix} X_1 \\ X_2 \\ Z \end{pmatrix}\} = \frac{\nu}{\nu - 2}\boldsymbol{\sigma}^2 \equiv \begin{pmatrix} 20 & 0 & 12 \\ 0 & 13.5 & 13 \\ 12 & 13 & 40 \end{pmatrix}.$$

   The optimal loadings are computed as

$$\boldsymbol{\beta} \equiv \mathbb{C}v\{\boldsymbol{X}, Z\}\mathbb{V}\{Z\}^{-1} = \left(\begin{smallmatrix} 12 \\ 13 \end{smallmatrix}\right) \times \frac{1}{40} = \left(\begin{smallmatrix} 0.3 \\ 0.325 \end{smallmatrix}\right).$$

   The shift parameters are computed as

$$\boldsymbol{\alpha} \equiv \mathbb{E}\{\boldsymbol{X}\} - \boldsymbol{\beta}\mathbb{E}\{Z\} = \left(\begin{smallmatrix} -2 \\ 2 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0.3 \\ 0.325 \end{smallmatrix}\right) \times 4 = \left(\begin{smallmatrix} -3.2 \\ 0.7 \end{smallmatrix}\right).$$

2. *[6 points]* Show that the linear factor model (2) is *not* idiosyncratic.

   The residuals have covariance matrix

   $$\mathbb{C}v\{\mathring{\boldsymbol{\varepsilon}}\} = \mathbb{C}v\{\boldsymbol{X}\} - \boldsymbol{\beta}\mathbb{C}v\{Z, \boldsymbol{X}\} = \begin{pmatrix} 20 & 0 \\ 0 & 13.5 \end{pmatrix} - \left(\begin{smallmatrix} 0.3 \\ 0.325 \end{smallmatrix}\right) \times \begin{pmatrix} 12 & 13 \end{pmatrix} = \begin{pmatrix} 16.4 & -3.9 \\ -3.9 & 9.275 \end{pmatrix}$$

3. *[6 points]* How would the results at the previous points change if the distribution of the target variables and factor were normal, with same location $\boldsymbol{\mu}$, and dispersion $2\boldsymbol{\sigma}^2$?

   Linear factor models are defined on an equivalence class of distributions with same expectation and covariance. Since the first two moments are the same, the results would be the same.

Solution 4 Homogeneous Markov chain for rating transitions

Consider a homogeneous Markov chain X_t that models the rating transitions of an obligor. The Markov chain has 8 states: {"AAA", "AA", "A", "BBB", "BB", "B", "CCC", "D"}, and the following one-year transition matrix

$$\boldsymbol{p} = \begin{pmatrix} 92.58\% & 6.36\% & 0.78\% & 0.23\% & 0.05\% & 0.00\% & 0.00\% & 0.00\% \\ 2.95\% & 92.15\% & 3.99\% & 0.73\% & 0.17\% & 0.01\% & 0.00\% & 0.00\% \\ 0.24\% & 4.60\% & 89.93\% & 4.60\% & 0.55\% & 0.06\% & 0.01\% & 0.01\% \\ 0.06\% & 0.50\% & 8.22\% & 87.10\% & 3.30\% & 0.54\% & 0.14\% & 0.14\% \\ 0.07\% & 0.26\% & 1.39\% & 10.72\% & 84.46\% & 2.62\% & 0.25\% & 0.23\% \\ 0.01\% & 0.03\% & 0.34\% & 2.11\% & 11.74\% & 81.97\% & 1.90\% & 1.90\% \\ 0.00\% & 0.01\% & 0.10\% & 1.61\% & 3.67\% & 7.10\% & 82.43\% & 5.08\% \\ 0.00\% & 0.00\% & 0.00\% & 0.00\% & 0.00\% & 0.00\% & 0.00\% & 100.00\% \end{pmatrix}. \quad (4)$$

1. *[6 points]* What properties does the transition probability matrix $\boldsymbol{p}$ satisfy?

   The properties of the matrix $\boldsymbol{p}$ are the following:
   - [probability] the rows must have positive entries that sum to one ($p_{r\to c} \geq 0$, $\sum_{c=1}^{\bar{c}} p_{r\to c} = 1$);
   - [default] default ("D" $\leftrightarrow \bar{c}$) is an absorbing state, hence the elements of the last row ($\bar{c}$) are all 0, except for the last one, which is 1 ($p_{\bar{c}\to\bar{c}} = 1$);
   - [monotonicity] the entries on each row decrease monotonically from the main diagonal, to reflect the fact that a transition to a farther state is less probable than a transition to a nearer state ($|r-c| \leq |r-c'| \Rightarrow p_{r\to c} \geq p_{r\to c'}$).

2. *[6 points]* Assume that the current rating is "BB". Compute the probability that the obligor, over the next year:
   a) stays in the same rating class;
   b) defaults;
   c) upgrades to a better rating.

   The required probabilities are the following:

a) The probability that, over the next year, the rating stays the same is $[\boldsymbol{p}]_{5,5} = 84.46\%$.
b) The probability that, over the next year, the obligor defaults is $[\boldsymbol{p}]_{5,8} = 0.23\%$
c) The probability that over the next year, the obligor upgrades to a better rating is $\sum_{c=1}^{4} [\boldsymbol{p}]_{5,c} = 12.44\%$

3. *[6 points]* Assume that the current rating is "BB". Suppose that we know the realizations of the invariant of the Markov chain for the next year is

$$\epsilon_{t+1} = 0.08.$$

What is the rating of the obligor after 1 year?

According to the next-step function of a Markov chain, the rating for the next year, given the realization $\epsilon_{t+1} = 0.08$, will be

$$x_{t+1} = q_X(0.08|r = 5) = \{c \text{ such that } 0.08 \in (F_X(c-1|r = 5), F_X(c|r = 5)]\}.$$

Since $\sum_{u=1}^{3} [\boldsymbol{p}]_{5,u} = 1.72\% < 0.08 < 12.44\% = \sum_{u=1}^{4} [\boldsymbol{p}]_{5,u}$, the obligor will be in state $c = 4$, i.e. the rating will be "BBB".