3.5 Case study: modeling the swap market

In this section we discuss how to model the swap market. Swaps are very liquid securities and many new contracts are traded every day. A υ-*swap* $(E-t)$-*forward* is a contract whose value at the generic time t reads:

$$P_t^{(E,\upsilon)} \equiv s\rho \sum_{k=1}^{\upsilon/\rho} Z_t^{(E_k)} + Z_t^{(E+\upsilon)} - Z_t^{(E)}. \tag{3.203}$$

In this formula s is the agreed upon *fixed rate* expressed in annualized terms: at inception t_0 this rate is typically set in such a way that the value of the contract zero, i.e. it is set as the $(E-t_0)$-into-υ forward par swap rate defined in (3.57); ρ is a fixed time-interval of the order of a few months; the generic term $E_k \equiv E + k\rho$ is one *fixed-leg payment date*; $Z_t^{(E)}$ is the price of a zero-coupon bond with maturity E. The pricing formula (3.203) originates from the structure of the contract, according to which agreed upon fixed payments are swapped against floating payments that depend on the current levels of interest rates, see Rebonato (1998) and Brigo and Mercurio (2001). Nevertheless, we can take (3.203) as the definition of a security.

In this case study the investment decision is taken at $T \equiv$ January 1st 2000 and we plan to invest in an "eight-year swap two-years forward", i.e. a swap that starts $(E-T) \equiv$ two years from the investment date on $E \equiv$ January 1st 2002 and ends $\upsilon \equiv$ eight years later on $E+\upsilon \equiv$ January 1st 2010. The fixed payments occur every $\rho \equiv$ three months. Therefore, this contract is determined by the price of thirty-three zero-coupon bonds.

We assume that the investment horizon is $\tau \equiv$ two months. Our aim is to determine the distribution of $P_{T+\tau}^{(E,\upsilon)}$. To do this, we dispose of the daily database of all the zero-coupon bond prices for the past ten years.

3.5.1 The market invariants

Everyday, many new forward swap contracts are issued with new starting and ending dates. Therefore, the swap market is completely priced by the set of all the zero-coupon bond prices for virtually all the maturities on a daily basis up to around thirty years in the future:

$$Z_t^{(E)} \text{ such that } E = t + 1d, t + 2d, \ldots, t + 30y. \tag{3.204}$$

The first step to model a market is to determine its invariants. We have seen in Section 3.1.2 that the natural invariants for the fixed-income market are the changes in yield to maturity:

$$X_{t,\widetilde{\tau}}^{(\upsilon)} \equiv Y_t^{(\upsilon)} - Y_{t-\widetilde{\tau}}^{(\upsilon)}. \tag{3.205}$$

In this expression $\widetilde{\tau}$ is the estimation interval and υ denotes a specific time to maturity in the *yield curve*, which is the plot of the yield to maturity as a function of the respective time to maturity:

$$\upsilon \mapsto Y_t^{(\upsilon)} \equiv -\frac{1}{\upsilon} \ln \left(Z_t^{(t+\upsilon)} \right), \quad \upsilon = 1d, 2d, \ldots, 30y. \tag{3.206}$$

If we were to invest in several swap contracts, due to the large number (3.204) of bonds involved in the swap market, we would need to model the joint distribution of the changes in yield to maturity for the whole yield curve. Nevertheless, even for our example of one swap contract we still need to model a big portion of the swap curve, namely the sector between two and ten years.

In our example we have access to the database of the prices of the zero-coupon bonds (3.204) every day for the past ten years. Equivalently, we have access to the whole yield curve (3.206) every day for the past ten years:

$$\upsilon \mapsto y_t^{(\upsilon)} \equiv -\frac{1}{\upsilon} \ln \left(z_t^{(t+\upsilon)} \right), \quad \begin{cases} \upsilon = 1d, 2d, \ldots, 30y \\ t = T - 10y, \ldots, T - 1d, T. \end{cases} \tag{3.207}$$

The lower case letters in this expression denote the realizations in the past of the respective random variables (3.206), which we denote with upper case letters.

We remark that in reality the zero-coupon bonds are not traded in the swap market and therefore their price is not directly available. Instead (3.203) represents the set of implicit equations, one for each swap contract, that define the prices of the underlying zero coupon bonds. The process of determining the zero-coupon prices from the prices of the swap contracts is called *bootstrapping*, see James and Webber (2000). This operation is performed by standard software packages.

To determine the distribution of the changes in the yield curve (3.205) we choose an estimation interval $\widetilde{\tau} \equiv$ one week, which presents a reasonable trade-off between the number of observations in the database and the reliability of the data with respect to the investment horizon. Indeed, the number of weekly observations from a ten-year sample exceeds five hundred. If we chose an estimation interval $\widetilde{\tau}$ equal to the investment horizon τ of two months, the number of observations in the dataset would be too small, i.e. about sixty observations for ten years of data. On the other hand, an estimation interval as short as, say, one day might give rise to spurious data and would not be suitable to extrapolate the distribution of the invariants at the investment horizon.

3.5.2 Dimension reduction

Consider the weekly invariants (3.205) relative to the section of the yield curve that prices our eight year swap two years forward:

$$X^{(\upsilon)} \equiv Y_t^{(\upsilon)} - Y_{t-\widetilde{\tau}}^{(\upsilon)}, \quad \upsilon = 2y, 2y + 1d, \ldots, 10y. \tag{3.208}$$

To ease the notation in this expression we dropped in the left hand side the specification of the estimation interval $\widetilde{\tau}$, which is fixed, and the dependence on time t, because the distribution of the invariants does not depend on t.

The invariants (3.208) constitute a set of a few thousand random variables. Therefore we need to reduce the dimension of the market invariants. In view of the principal component analysis approach to dimension reduction we focus on the covariance matrix of the weekly invariants:

$$C(v,p) \equiv \mathrm{Cov}\left\{X^{(v)}, X^{(v+p)}\right\}. \tag{3.209}$$

The following intuitive relations can be checked with the data.

The covariance matrix is a smooth function of the times to maturity in both directions. For example, the covariance of the three-year rate with the five-year rate is very close to both the covariance of the three-year rate with the five-year-plus-one-day rate and to the covariance of the three-year-plus-one-day rate with the five-year rate. Therefore:

$$C(v,p+dv) \approx C(v+dv,p), \tag{3.210}$$

which means that C is a smooth function of its arguments.

The diagonal elements of the covariance matrix, i.e. the variances of the rate changes at the different maturities, are approximately similar. For example, if borrowing money for three years becomes all of a sudden more expensive, so does borrowing money for ten years, and the change is approximately similar. Therefore:

$$C(v,0) \approx C(v+\tau,0). \tag{3.211}$$

In particular, the correlation matrix is approximately proportional to the covariance matrix.

The correlation of equally spaced times to maturity is approximately the same. For example, the correlation of the one-year rate with the two-year rate is approximately similar to the correlation of the four-year rate with the five-year rate. Therefore:

$$C(v,p) \approx C(v+\tau,p). \tag{3.212}$$

The correlation matrix decreases away from the diagonal. For example, the correlation of the one-year rate with the two-year rate is less than the correlation of the one-year rate with the five-year rate.

From the above properties we derive that the covariance matrix, in addition to being symmetric and positive, has the following approximate structure:

$$C(v,p) \approx h(p), \tag{3.213}$$

where h is a smooth, positive and decreasing function that is symmetrical around the origin:

$$h(p) = h(-p). \tag{3.214}$$

A matrix with this structure is called a *Toeplitz matrix*, see Figure 3.16. Therefore, the covariance matrix is a symmetric and positive smooth Toeplitz matrix that decays to zero away from the diagonal.

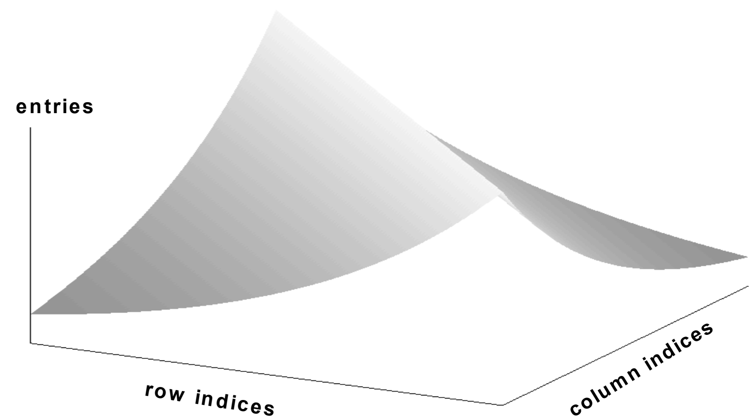

Fig. 3.16. Toeplitz matrix

The continuum limit

Although our ultimate purpose is to reduce the dimension of the swap market invariants, to gain more insight into the structure of randomness of this market we start looking in the opposite direction, namely the infinite-dimensional, continuum limit of the yield curve.

Indeed, the set of possible times to maturity is so dense, i.e. every day from two to ten years, that we can consider this parameter as a continuum: $\upsilon \in [2, 10]$. Therefore we consider the yield curve $Y_t^{(\upsilon)}$ as a stochastic object parametrized by two continuous indices: time t and time to maturity υ.

We now perform the principal component analysis (PCA) of the yield curve and then perform the PCA dimension reduction discussed in Section 3.4.2. We recall from (3.149) that the PCA decomposition of the covariance matrix of the invariants reads:

$$\operatorname{Cov}\{\mathbf{X}\}\,\mathbf{e}^{(n)} = \lambda_n \mathbf{e}^{(n)}, \tag{3.215}$$

for each eigenvector $\mathbf{e}^{(n)}$ and each eigenvalue λ_n, $n = 1, \ldots, N$.

By means of the analogies in Table B.4, Table B.11 and Table B.20, in the continuum (3.215) becomes the following spectral equation:

$$\int_{\mathbb{R}} \operatorname{Cov}\left\{X^{(\upsilon)}, X^{(p)}\right\} e^{(\omega)}(p)\, dp = \lambda_\omega e^{(\omega)}(\upsilon), \tag{3.216}$$

where λ_ω is the generic eigenvalue and $e^{(\omega)}$ is the *eigenfunction* relative to that eigenvalue.

We prove in Appendix www.3.6 that the generic eigenfunction of a Toeplitz covariance matrix must be an oscillating function with frequency ω, modulo a multiplicative factor:

$$e^{(\omega)}(\upsilon) \equiv e^{i\omega\upsilon}, \quad \omega \in [0, +\infty). \tag{3.217}$$

These eigenfunctions determine the directions of the principal axes of the infinite-dimensional location-dispersion ellipsoid of our invariants, refer to Figure 3.14 for the finite-dimensional case.

Furthermore, the generic eigenvalue λ_ω is the Fourier transform (B.34) evaluated at the frequency ω of the cross-diagonal function (3.213) that determines the structure of the covariance matrix:

$$\lambda_\omega = \mathcal{F}[h](\omega). \tag{3.218}$$

Consider now the invariants (3.208) and suppose that we perform the dimension reduction by means of PCA as in (3.160), i.e. we project the original invariants onto the hyperplane spanned by a few among the principal axes of the location-dispersion ellipsoid:

$$X^{(\upsilon)} \mapsto \widetilde{X}^{(\upsilon)}, \quad \upsilon \in [2, 10]. \tag{3.219}$$

This corresponds to selecting a subset Ω of all the possible frequencies. To evaluate the quality of the PCA dimension reduction, we compute the generalized r-square (3.162), which in this context reads:

$$R^2\left\{X, \widetilde{X}\right\} \equiv \frac{\int_\Omega \lambda_\omega d\omega}{\int_0 \lambda_\omega d\omega}. \tag{3.220}$$

In order to select the best frequencies Ω we need to assign a parametric form to the cross-diagonal function (3.213) that determines the structure of the covariance matrix. To this purpose, we introduce the concept of *string*, or *random field*, see James and Webber (2000): a string is a stochastic object parametrized by two continuous indices. In our case, the yield curve is a string, and the market invariants, i.e. the changes in yield to maturity (3.208), become the discrete increment of a random field along the time dimension. A result in Kennedy (1997) states that, under fairly general conditions, the covariance of a family of invariants that stem from a random field has the following structure:

$$\operatorname{Cov}\left\{X^{(\upsilon)}, X^{(\upsilon+p)}\right\} = \sigma^2 e^{-\mu\upsilon} e^{-\gamma|p|}, \tag{3.221}$$

where $\mu \geq 0$ and $\gamma \geq \mu/2$.

Indeed, this is the structure of the covariance matrix that we have derived in (3.213), where $\mu \approx 0$. In other words, the covariance of the weekly changes in yield to maturity in the continuum limit has the following cross-diagonal functional form:

$$h(p) = \sigma^2 \exp(-\gamma|p|). \tag{3.222}$$

Substituting this expression in (3.218) we obtain the explicit form of the eigenvalues:[4]

[4] The reader might notice that this principal component analysis is the spectral analysis of a one-dimensional Ornstein-Uhlenbeck process.

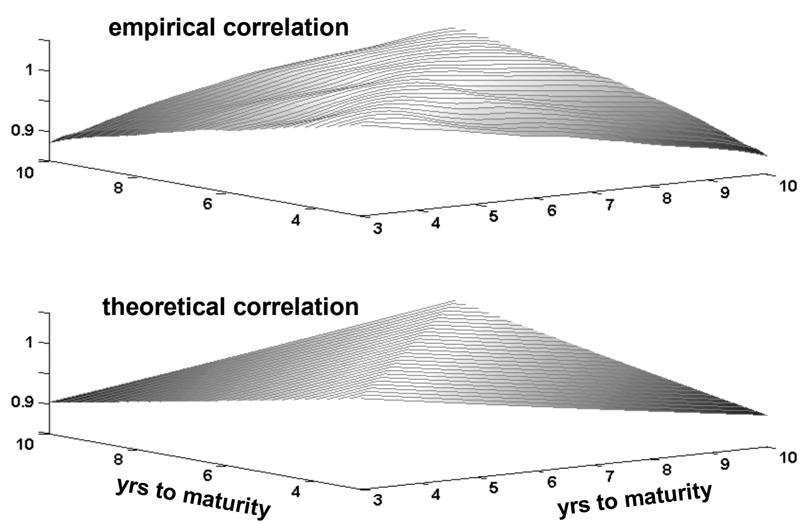

Fig. 3.17. Correlations among changes in interest rates

$$\lambda_\omega = \frac{2\sigma^2}{\sqrt{\gamma^2}} \left(1 + \frac{\omega^2}{\gamma^2}\right)^{-1}, \tag{3.223}$$

see Appendix www.3.6. Notice that the eigenvalues decrease with the frequency. Indeed, (3.223) is the *Lorentz function*, which is proportional to the probability density function of a Cauchy distribution centered in the origin $\omega \equiv 0$ with dispersion parameter γ, see (1.79). In other words, the set of preferred frequencies reads:

$$\Omega \equiv [0, \overline{\omega}], \tag{3.224}$$

for some cut-off value $\overline{\omega}$.

To choose the proper cut-off, we fit the theoretical expression of the correlation matrix, which we obtain from (3.222), to the empirical correlation matrix:

$$\text{Cor}\left\{X^{(\upsilon)}, X^{(\upsilon+p)}\right\} = \exp\left(-\gamma\,|p|\right). \tag{3.225}$$

In the swap market, measuring time to maturity in years, we obtain for γ the numerical value $\gamma \approx 0.0147$. Such a low value of the parameter corresponds to highly correlated changes in yield, see Figure 3.17. Since the parameter γ in (3.225) is small, i.e. the changes in yield at different times to maturity are highly correlated, the eigenvalues (3.223) decrease sharply to zero as the respective frequency moves away from the origin. We plot this profile in the top portion of Figure 3.18. This situation is typical of the Fourier transform: when a function decays slowly to infinity, its Fourier transform decays fast, and viceversa, as we see for instance in $(B.37)$. At an extreme, the Fourier transform of the Dirac delta centered in zero ($B.38$) is a constant, i.e. a flat function.

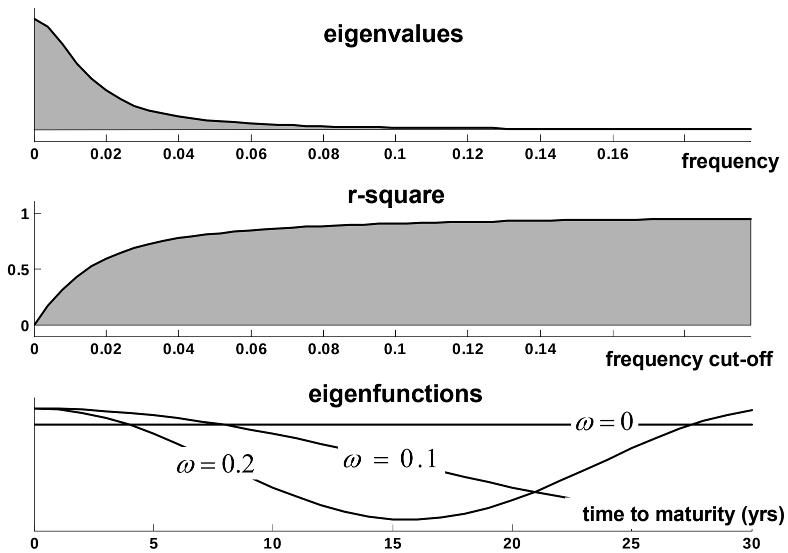

Fig. 3.18. Swap curve PCA: the continuum limit

Therefore, we expect the lowest frequencies to recover almost all of the randomness in the swap market. To quantify this more precisely, we compute the generalized r-square (3.220) obtained by considering only the lowest frequencies. A simple integration of (3.223) yields the following analytical expression:

$$R^2\left\{X, \widetilde{X}\right\} \equiv \frac{\int_0^{\overline{\omega}} \lambda_\omega d\omega}{\int_0^{+\infty} \lambda_\omega d\omega} = \frac{2}{\pi} \arctan\left(\frac{\overline{\omega}}{\gamma}\right). \tag{3.226}$$

In the middle portion of Figure 3.18 we display the generalized r-square (3.226) as a function of the cut-off frequency $\overline{\omega}$: the bulk of randomness is recovered by frequencies lower than $\overline{\omega} \approx 0.2$.

In the bottom portion of Figure 3.18 we see that this frequency corresponds to eigenfunctions that complete an oscillation over a thirty-year period. This is the span of time-to-maturities covered by the swap market.

To summarize, we draw the following lesson from the continuum-limit of the swap market. Since changes in interest rates are highly correlated, we can reduce the dimension of randomness in the swap market by considering a limited number of directions of randomness, i.e. those directions defined by the eigenfunctions that oscillate less than once within the set of time to maturities considered.

The discrete and finite case

Given the high correlation among changes in yield at adjacent times to maturity, instead of considering a continuous yield curve, we can safely consider time to maturities one year apart and implement the PCA dimension reduction on this discrete set, see Litterman and Scheinkman (1991).

In our example, this step shrinks the dimension from infinity to nine:

$$\mathbf{X} \equiv \left(X^{(2y)}, \ldots, X^{(10y)}\right)'. \tag{3.227}$$

First of all we estimate the 9×9 covariance matrix as in Chapter 4. Then we perform the PCA decomposition (3.149) of the covariance matrix:

$$\mathrm{Cov}\{\mathbf{X}\} = \mathbf{E}\boldsymbol{\Lambda}\mathbf{E}'. \tag{3.228}$$

In this expression $\boldsymbol{\Lambda}$ is the diagonal matrix of the $N \equiv 9$ eigenvalues of the covariance sorted in decreasing order:

$$\boldsymbol{\Lambda} \equiv \mathrm{diag}(\lambda_1, \ldots, \lambda_9); \tag{3.229}$$

and the matrix $\mathbf{E}$ is the juxtaposition of the respective eigenvectors and represents a rotation:

$$\mathbf{E} \equiv \left(\mathbf{e}^{(1)}, \ldots, \mathbf{e}^{(9)}\right). \tag{3.230}$$

At this point we are ready to reduce the dimension further. First we *define* all the potential $N \equiv 9$ factors:

$$\mathbf{F} \equiv \mathbf{E}'(\mathbf{X} - \mathrm{E}\{\mathbf{X}\}). \tag{3.231}$$

Then we decide the number K of factors to consider for the dimension reduction, see (3.159).

In the top portion of Figure 3.19 we plot the eigenvalues (3.229). The first few eigenvalues overwhelmingly dominate the others. This becomes more clear in the middle portion of Figure 3.19, where we draw the generalized r-square of the first K factors (3.162) as a function of K. We see that the first three factors account for 99% of the total randomness. Therefore we set $K \equiv 3$, i.e. we consider as factors the first three entries of (3.231).

Since the invariants (3.208) are the changes in yield to maturity:

$$\mathbf{X} \equiv \mathbf{Y}_t - \mathbf{Y}_{t-\widetilde{\tau}}, \tag{3.232}$$

and since the expected value of the yield curve is approximately the previous realization of the yield curve:

$$\mathrm{E}\{\mathbf{Y}_t\} \approx \mathbf{Y}_{t-\widetilde{\tau}} \Leftrightarrow \mathrm{E}\{\mathbf{X}\} \approx \mathbf{0}, \tag{3.233}$$

recovering the invariants as in (3.160) corresponds to recovering the following yield curve at time t from the yield curve realized at time $t - \widetilde{\tau}$:

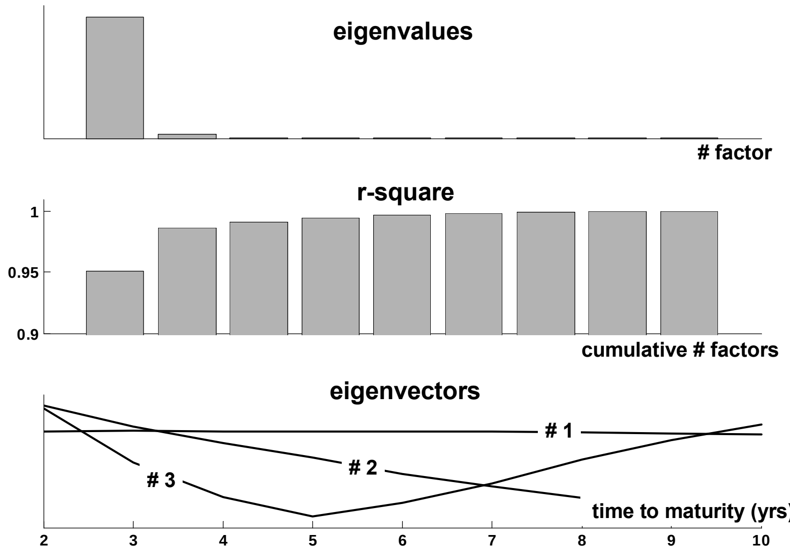

Fig. 3.19. Swap curve PCA: the discrete case

$$\widetilde{\mathbf{Y}}_t \equiv \mathbf{Y}_{t-\widetilde{\tau}} + \mathbf{e}^{(1)} F_1 + \mathbf{e}^{(2)} F_2 + \mathbf{e}^{(3)} F_3. \tag{3.234}$$

Consider the first factor. From (3.163) the square root of the first eigenvalue is the standard deviation of the first factor:

$$\mathrm{Sd}\{F_1\} = \sqrt{\lambda_1}. \tag{3.235}$$

Therefore a one-standard-deviation event in the first factor moves the curve as follows:

$$\mathbf{Y}_t \mapsto \mathbf{Y}_{t+\widetilde{\tau}} \equiv \mathbf{Y}_t \pm \sqrt{\lambda_1}\mathbf{e}^{(1)}, \tag{3.236}$$

where $\mathbf{e}^{(1)}$ is the first eigenvector. The plot of this eigenvector in the bottom portion of Figure 3.19 shows that the first eigenvector is approximately a positive constant. In a geometrical interpretation, the longest principal axis of the location-dispersion ellipsoid pierces the positive octant, see Figure 3.20. This happens because, as in (3.222), the elements of the covariance matrix of the changes in yield to maturity are positive, and thus the Perron-Frobenius theorem applies, see Appendix A.5.

Therefore the one-standard-deviation event in the first factor (3.236) corresponds to a *parallel shift* of the curve. In the top portion of Figure 3.21 we plot a three-standard-deviation parallel shift. From the estimation we obtain that this corresponds to the following change:

$$3\sqrt{\lambda_1} e_k^{(1)} \approx 3 \times 42 \times .33 \approx 42 \text{ b.p.}, \tag{3.237}$$

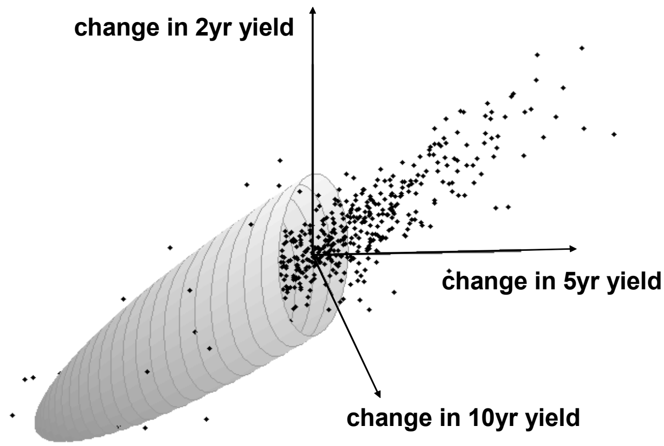

Fig. 3.20. Swap curve PCA: location-dispersion ellipsoid fitted to observations

where "b.p." stands for *basis point*. The basis point is a unit of measure for interest rates which is equal to 1/10000: in other words, an $r\%$ interest rate is equal to $100r$ b.p..

Consider now the second factor. Similarly to the first eigenvalue, from (3.163) the second eigenvalue is the variance of the second factor. Therefore a one-standard-deviation event in the second factor moves the curve as follows:

$$\mathbf{Y}_t \mapsto \mathbf{Y}_{t+\widetilde{\tau}} \equiv \mathbf{Y}_t \pm \sqrt{\lambda_2}\mathbf{e}^{(2)}, \tag{3.238}$$

where $\mathbf{e}^{(2)}$ is the second eigenvector, which we display in the bottom portion of Figure 3.19. This eigenvector is a decreasing line: thus a one-standard-deviation event in the second factor corresponds to a *steepening/flattening* of the curve. We remark that $\lambda_2 \ll \lambda_1$, whereas the entries of $\mathbf{e}^{(2)}$ are of the same order as those of $\mathbf{e}^{(1)}$ (because both vectors have length equal to one by construction). Therefore a one-standard-deviation event in the second factor has a much smaller influence on the yield curve than a one-standard-deviation event in the first factor. In the middle portion of Figure 3.21 we plot a three-standard-deviation steepening/flattening.

Finally, consider the third factor. From (3.163) the third eigenvalue is the variance of the third factor. Therefore a one-standard-deviation event in the third factor moves the curve as follows:

$$\mathbf{Y}_t \mapsto \mathbf{Y}_{t+\widetilde{\tau}} \equiv \mathbf{Y}_t \pm \sqrt{\lambda_3}\mathbf{e}^{(3)}, \tag{3.239}$$

where $\mathbf{e}^{(3)}$ is the third eigenvector, which we display in the bottom portion of Figure 3.19. This eigenvector is hump-shaped: therefore a one-standard-

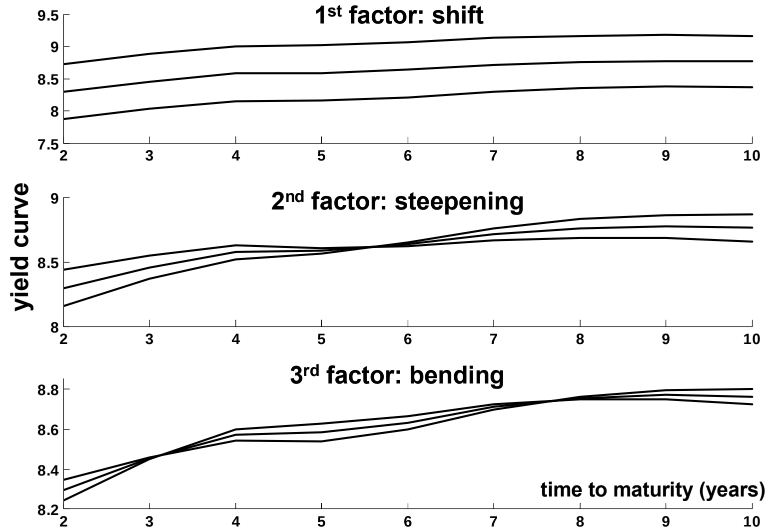

Fig. 3.21. Three-standard-deviation effects of PCA factors on swap curve

deviation event in the third factor corresponds to a *curvature effect* on the yield curve. We remark that $\lambda_3 \ll \lambda_2 \ll \lambda_1$, whereas the entries of $\mathbf{e}^{(3)}$ are of the same order as those of $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$. Therefore a one-standard-deviation event in the third factor has a much smaller influence on the yield curve than a one-standard-deviation event in either the first or the second factor. In the bottom portion of Figure 3.21 we plot a three-standard-deviation curvature.

Looking back, we have managed to reduce the dimension of the two-to-ten-year section of the swap market from an infinite number of factors, as summarized in Figure 3.18, to three factors only, as summarized in Figure 3.19 with an accuracy of 99%. The reader is invited to ponder on the many analogies between these two figures.

3.5.3 The invariants at the investment horizon

We have reduced above the sources of randomness in the swap market to only three hidden factors that explain 99% of the total yield curve changes. Here we model the distribution of these three factors for the estimation period, which in our example is one week, and project it to the investment horizon, which in our case is two month.

First, we estimate the joint distribution of the three factors which from (3.231) and (3.233) are defined as follows:

$$(F_1, F_2, F_3) \equiv \mathbf{X}' \left(\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \mathbf{e}^{(3)}\right). \tag{3.240}$$

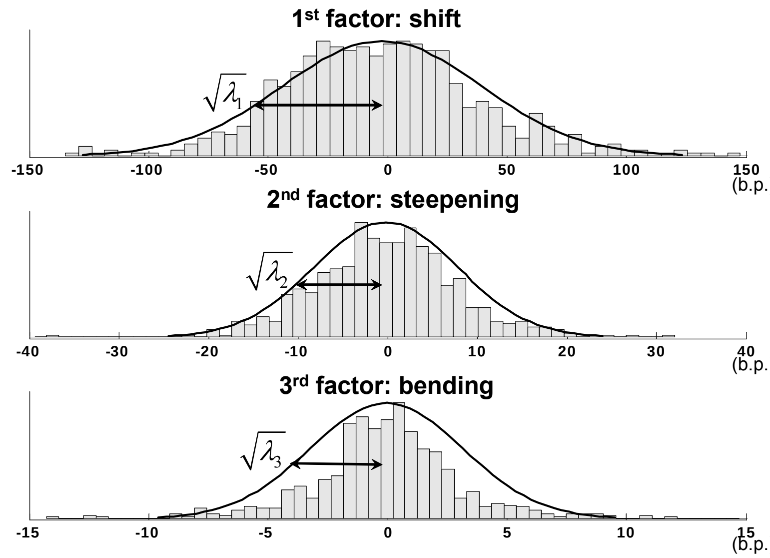

Fig. 3.22. Marginal distribution of swap curve PCA factors

In Figure 3.22 we plot the histogram of the observed values of each factor, which is a proxy for their respective marginal distributions. We model the joint distribution of the factors as a normal distribution:

$$\mathbf{F}_{t,\widetilde{\tau}} \sim \mathrm{N}\left(\mathbf{0}, \mathrm{diag}\left(\lambda_1, \lambda_2, \lambda_3\right)\right), \tag{3.241}$$

where in the notation we stressed that the distribution of the factors refers to the estimation interval $\widetilde{\tau}$, which in our case is one week.

To project the distribution of the invariants to the investment horizon τ we make use of (3.64). Performing the same steps as in Example (3.74) we obtain that the factors at the investment horizon are normally distributed with the following parameters:

$$\mathbf{F}_{T+\tau,\tau} \sim \mathrm{N}\left(\mathbf{0}, \frac{\tau}{\widetilde{\tau}}\,\mathrm{diag}\left(\lambda_1, \lambda_2, \lambda_3\right)\right). \tag{3.242}$$

In our case the investment horizon is two months ahead, and thus $\tau/\widetilde{\tau} \approx 8$.

We stress that here we are neglecting estimation risk. In other words, the distribution at the investment horizon is given precisely by (3.242) *if* the estimation-horizon distribution is precisely (3.241). Nevertheless, in the first place we used here a very rough estimation/fitting process: we will discuss the estimation of the market invariants in detail in Chapter 4. Secondly, no matter how good the estimate, an estimate is only an approximation to reality, and thus the distribution at the investment horizon cannot be precise. In fact, the farther in the future the investment horizon, the larger the effect of the estimation error. We discuss estimation risk and how to cope with it extensively in the third part of the book.

3.5.4 From invariants to prices

From the pricing function at the generic time t of the generic forward swap contract (3.203) and the expression of the zero-coupon bond prices in terms of the invariants (3.81), we obtain the pricing formula at the investment horizon in the form (3.79) of the swap in terms of the investment-horizon market invariants (3.205), i.e. the changes in yield to maturity from the investment decision to the investment horizon:

$$P_{T+\tau}^{(E,\upsilon)} = g(\mathbf{X}) \equiv \sum_{k=0}^{\upsilon/\rho} c_k Z_T^{(\upsilon_k)} e^{-X_{T+\tau,\tau}^{(\upsilon_k)} \upsilon_k}. \tag{3.243}$$

We recall that in this expression the coefficients c_k are defined in terms of the agreed upon rate s and the interval ρ between fixed payments as follows:

$$c_0 \equiv -1, \quad c_{\upsilon/\rho} \equiv 1 + s\rho, \quad c_k \equiv s\rho, \quad k = 1, \ldots, \frac{\upsilon}{\rho} - 1; \tag{3.244}$$

and that the set of times to maturities read:

$$\upsilon_k \equiv E + k\rho - (T + \tau), \qquad k = 0, \ldots, \frac{\upsilon}{\rho}. \tag{3.245}$$

We stress that these are the times to maturity at the investment horizon, *not* at the time the investment decision is made.

To compute the distribution of the value of the forward swap contract at the investment horizon (3.243) we can take two routes.

On the one hand, since (3.243) is a sum of log-distributions, by means of (3.93) we can compute all the cross-moments of the terms in the sum (3.243) and from these and (2.93) we can compute all the moments of the distribution of the swap contract.

Alternatively, we can obtain quick and intuitive results by means of a series of approximations. Here we choose this second option.

To evaluate the goodness of the approximations to come, we also compute the exact distribution numerically: we simulate a large number of invariant scenarios by means of the three-factor model (3.242) and we apply the exact pricing formula (3.243) for all the above scenarios. In the bottom portion of Figure 3.23 we plot the histogram of the simulation, which represents the profile of the probability density function of the distribution of (3.243). We also plot the value at the time the investment decision is made, in view of evaluating the risk/reward profile of our investment.

- **Approximation 1: one factor**

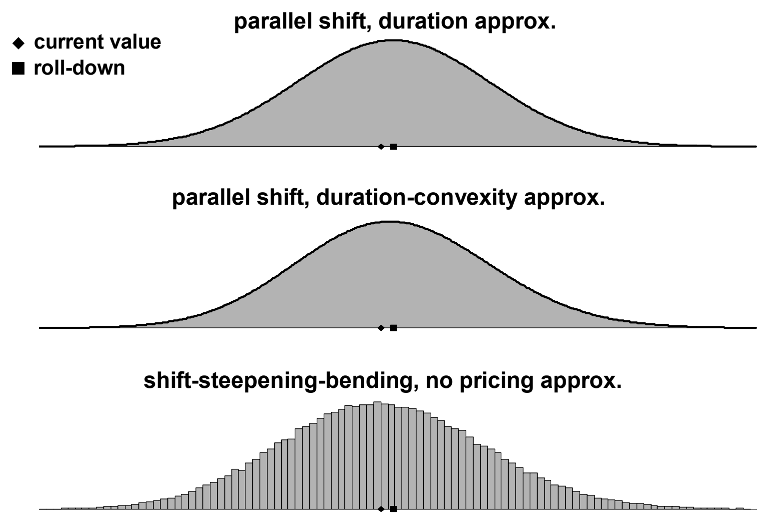

Fig. 3.23. Swap price distribution at the investment horizon

In view of performing an approximation, we do not need to consider three factors in the dimension reduction process discussed in Section 3.5.2. As we see in Figure 3.19 the first factor already explains 95% of the randomness in the swap market. Therefore, we focus on one factor only which from (3.242) has the following distribution:

$$F_{T+\tau,\tau} \sim \mathrm{N}\left(0, \frac{\tau}{\widetilde{\tau}}\lambda_1\right). \tag{3.246}$$

- **Approximation 2: parallel shift**

Without loss of generality, we can always rescale the first factor loading by a positive constant as follows:

$$\mathbf{e}^{(1)} \mapsto \alpha\mathbf{e}^{(1)}, \tag{3.247}$$

as long as we rescale the first factor accordingly, along with the distribution (3.246), in such a way that the effect of the factor on the curve (3.236) remains unaltered:

$$F \mapsto \frac{F}{\alpha} \quad \Rightarrow \quad \lambda \mapsto \frac{\lambda}{\alpha^2}. \tag{3.248}$$

We see in Figure 3.19 that the first eigenvector is almost "flat". Therefore, we can choose α in such a way that

$$\alpha\mathbf{e}^{(1)} \approx \mathbf{1}, \tag{3.249}$$

where **1** is a vector of ones.

The second approximation consists in assuming that (3.249) is exact. This implies that, due to the influence of the first factor (3.236), at each of the nodes corresponding to (3.245) the curve moves in exactly a parallel way and thus all the changes in yield coincide:

$$X_{T+\tau,\tau}^{(\upsilon_k)} \equiv X, \quad k = 0, \ldots, \frac{\upsilon}{\rho}. \tag{3.250}$$

From (3.246) the distribution of this common shift is:

$$X \sim \mathrm{N}\left(\mu, \sigma^2\right), \tag{3.251}$$

where

$$\mu \equiv 0, \quad \sigma^2 \equiv \frac{\tau}{\widetilde{\tau}} \frac{\lambda_1}{\alpha^2}. \tag{3.252}$$

From Figure 3.19 we realized that this second approximation can deteriorate the generalized r-square at most by a few percentage points.

- **Approximation 3: Taylor expansion**

We can consider the swap contract as a derivative product and perform as in (3.108) a Taylor expansion approximation of the exact pricing function (3.243) around zero, which from (3.233) represents the expected value of the invariants.

The order zero coefficient in the Taylor expansion of the value of the swap at the investment horizon (3.243) is called the *roll-down*. The roll-down is the value of the swap at the investment horizon if the invariant is zero, i.e. if the yield curve remains unchanged:

$$\mathrm{RD}(T, \tau) \equiv g(\mathbf{0}) = \sum_{k=0}^{\upsilon/\rho} c_k Z_T^{(\upsilon_k)}. \tag{3.253}$$

The (opposite of the) first-order coefficient in the Taylor expansion of the value of the swap at the investment horizon (3.243) is called the *present value of a basis point* (PVBP):

$$\mathrm{PVBP}(T, \tau) \equiv -\sum_{k=0}^{\upsilon/\rho} \partial_k g|_{\mathbf{X}=\mathbf{0}} = \sum_{k=0}^{\upsilon/\rho} \upsilon_k c_k Z_T^{(\upsilon_k)}. \tag{3.254}$$

The PVBP is a weighted sum of the times to maturity at the investment horizon (3.245) of the zero-coupon bonds involved in the pricing formula of the swap contract. We mention that practitioners in the (very similar) bond market consider the first-order term normalized by the roll-down, which is called the *duration* of the bond and has the dimensions of time.

The second-order coefficient in the Taylor expansion of the value of the swap at the investment horizon (3.243) is called the *convexity adjustment*:

$$\mathrm{Conv}\,(T,\tau) \equiv \sum_{k,j=0}^{\upsilon/\rho} \partial_{jk} g|_{\mathbf{X}=\mathbf{0}} = \sum_{k=0}^{\upsilon/\rho} c_k \upsilon_k^2 Z_T^{(\upsilon_k)}. \tag{3.255}$$

In standard contracts the convexity is typically positive: the the only negative term in the sum (3.255) is the one corresponding to $k \equiv 0$ and is typically outweighed by the other terms.

From the definitions of roll-down, PVBP and convexity, the value at the investment horizon of the swap (3.243) and the relation (3.250) we obtain the following second-order Taylor approximation:

$$P_{T+\tau}^{(E,\upsilon)} \approx \mathrm{RD} - \mathrm{PVBP}\, X + \frac{1}{2}\,\mathrm{Conv}\, X^2 + \cdots. \tag{3.256}$$

If we stop at the order zero in (3.256) we obtain a value, the roll-down, which is different than the value of the swap at the time the investment decision is made, see Figure 3.23. This difference is known as the *slide* of the contract. Some traders try to "cheat the curve", investing based on the (dis)advantages of the roll-down.

If we stop at the first order in (3.256) the distribution at the investment horizon of our swap contract is linear in the invariant. From (3.251) we obtain its distribution:

$$P_{T+\tau}^{(E,\upsilon)} \sim \mathrm{N}\left(\mathrm{RD}, \sigma^2\, \mathrm{PVBP}^2\right). \tag{3.257}$$

In the top plot of Figure 3.23 we display the probability density function of (3.257) as well as the current value and the roll-down.

If we stop at the second order in (3.256), rearranging the terms we can rewrite the swap value at the investment horizon as follows:

$$P_{T+\tau}^{(E,\upsilon)} = c + W^2, \tag{3.258}$$

where c is a constant defined as follows:

$$c \equiv \mathrm{RD} - \frac{1}{2}\frac{\mathrm{PVBP}^2}{\mathrm{Conv}}, \tag{3.259}$$

and W is the following normal random variable:

$$W \equiv \sqrt{\frac{\mathrm{Conv}}{2}} X - \frac{\mathrm{PVBP}}{\sqrt{2\,\mathrm{Conv}}}. \tag{3.260}$$

Therefore the second-order approximation of the swap contract has a shifted non-central gamma distribution with one degree of freedom and the following non-centrality and scale parameters:

$$P_{T+\tau}^{(E,\upsilon)} \sim \mathrm{Ga}\left(1, \sqrt{\frac{\mathrm{Conv}}{2}}\mu - \frac{\mathrm{PVBP}}{\sqrt{2\,\mathrm{Conv}}}, \frac{\mathrm{Conv}}{2}\sigma^2\right), \tag{3.261}$$

see (1.107). It is convenient to represent this distribution in terms of its characteristic function, which reads:

$$\begin{aligned}\phi_P(\omega) = &\left(1 - i\omega \operatorname{Conv} \sigma^2\right)^{-\frac{1}{2}} e^{-\frac{1}{2}\frac{(\operatorname{Conv}\mu - \mathrm{PVBP})^2 \sigma^2}{1 - i\omega \operatorname{Conv}\sigma^2}} \\ &e^{i\omega\left(\mathrm{RD} - \mathrm{PVBP}\,\mu + \frac{1}{2}\operatorname{Conv}\mu^2\right)}.\end{aligned} \tag{3.262}$$

This is a specific instance of (5.30), a result which we prove and discuss later in a more general context.

In the middle plot of Figure 3.23 we display the probability density function of (3.258) as well as the current value and the roll-down. From a comparison of the three plots in Figure 3.23 we see that the simple parallel shift/duration approximation provides very satisfactory results.