5

Evaluating allocations

An allocation is a portfolio of securities in a given market. In this chapter we discuss how to evaluate an allocation for a given investment horizon, i.e. a linear combination of the prices of the securities at the investment horizon.

In Section 5.1 we introduce the investor's objectives. An objective is a feature of a given allocation on which the investor focuses his attention. For instance an objective is represented by final wealth at the horizon, or net gains, or wealth relative to some benchmark. The objective is a random variable that depends on the allocation. Although it is not possible to compute analytically the distribution of the objective in general markets, we present some approximate techniques that yield satisfactory results in most applications.

In Section 5.2 we tackle the problem of evaluating allocations, or more precisely the distribution of the objective relative to a given allocation. We do this by introducing the concept of stochastic dominance, a criterion that allows us to evaluate the distribution of the objective as a whole: when facing two allocations, i.e. the distributions of two different objectives, the investor will choose the one that is more advantageous in a global sense. Nevertheless, stochastic dominance presents a few drawbacks, most notably the fact that two generic allocations might not necessarily be comparable. In other words, the investor might not be able to rank allocations and thus make a decision regarding his investment.

As a consequence, in Section 5.3 we take a different approach. We summarize all the properties of a distribution in a single number: an index of satisfaction. If the index of satisfaction is properly defined the investor can in all circumstances choose the allocation that best suits him. Therefore we analyze a set of criteria that a proper satisfaction index should or could satisfy, such as estimability, consistency with stochastic dominance, constancy, homogeneity, translation invariance, additivity, concavity, risk aversion.

In the remainder of the chapter we discuss three broad classes of indices of satisfaction that have become popular among academics and practitioners.

In Section 5.4 we present the first of such indices of satisfaction: the certainty-equivalent. Based on the intuitive concept of expected utility, this has been historically the benchmark criterion to assess allocations. After introducing the definition of the certainty-equivalent and discussing its general properties, we show how to build utility functions that cover a wide range of situations, including the non-standard setting of prospect theory. Then we tackle some computational issues. Indeed, the computation of the certainty-equivalent involves integrations and functional inversions, which are in general impossible to perform. Therefore we present some approximate results, such as the Arrow-Pratt expansion. Finally, we perform a second-order sensitivity analysis to determine the curvature of the certainty-equivalent. The curvature is directly linked to the investor's attitude toward diversification and it is fundamental in view of computing numerical solutions to allocation problems.

In Section 5.5 we consider another index of satisfaction, namely the quantile of the investor's objective for a given confidence level. This index is better known under the name of value at risk when the investor's objective are net gains. The quantile-based index of satisfaction has become a standard tool among practitioners after the Basel Accord enforced its use among financial institutions to monitor the riskiness of their investment policies. After introducing the definition of the quantile-based index of satisfaction and discussing its general properties, we tackle some computational issues. Approximate expressions of the quantile can be obtained with approaches such as the Cornish-Fisher expansion and extreme value theory. Finally, we perform a second-order sensitivity analysis, from which it follows that quantile-based indices of satisfaction fail to promote diversification.

In Section 5.6 we discuss a third group of measures of satisfaction: coherent indices and spectral indices, which represent a sub-class of coherent indices. These measures of satisfaction are defined axiomatically in terms of their properties, most notably the fact that by definition they promote diversification. Nevertheless, spectral indices of satisfaction can also be introduced alternatively as weighted averages of a very popular measure of risk, the expected shortfall. This representation is more intuitive and suggests how to construct coherent indices in practice. As we did for the certainty-equivalent and the quantile, we discuss the computational issues behind the spectral indices of satisfaction. Finally, we perform a second-order sensitivity analysis. In particular, from this analysis it follows that spectral measures of satisfaction are concave and thus promote diversification.

We remark that throughout the chapter all the distributions are assumed continuous and smooth, possibly after regularizing them as discussed in Appendix B.4.

5.1 Investor's objectives

Consider a market of N securities. At the time T when the investment is made the investor can purchase α_n units of the generic n-th security. These units are specific to the security: for instance, in the case of equities the units are shares, in the case of futures the units are contracts, etc. Therefore, the *allocation* is represented by the N-dimensional vector $\boldsymbol{\alpha}$.

We denote as $P_t^{(n)}$ the price at the generic time t of the generic n-th security. With the allocation $\boldsymbol{\alpha}$ the investor forms a portfolio whose value at the time the investment decision is made is:

$$w_T(\boldsymbol{\alpha}) \equiv \boldsymbol{\alpha}'\mathbf{p}_T, \tag{5.1}$$

where the lower-case notation emphasizes that the above quantities are known at the time the investment decision is made.

At the investment horizon τ the market prices of the securities are a multivariate random variable. Therefore at the investment horizon the portfolio is a one-dimensional random variable, namely the following simple function of the market prices:

$$W_{T+\tau}(\boldsymbol{\alpha}) \equiv \boldsymbol{\alpha}'\mathbf{P}_{T+\tau}. \tag{5.2}$$

The investor has one or more *objectives* Ψ, namely quantities that the investor perceives as beneficial and therefore he desires in the largest possible amounts. This is the *non-satiation principle* underlying the investor's objectives. The standard objectives are discussed below.

- **Absolute wealth**

The investor focuses on the value at the horizon of the portfolio:

$$\Psi_{\boldsymbol{\alpha}} \equiv W_{T+\tau}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}'\mathbf{P}_{T+\tau}. \tag{5.3}$$

For example, personal financial planning focuses on total savings. Therefore for the private investor who makes plans on his retirement, the horizon is of the order of several years and the objective is the final absolute wealth at his investment horizon.

- **Relative wealth**

The investor is concerned with overperforming a reference portfolio, whose allocation we denote as $\boldsymbol{\beta}$. Therefore the objective is:

$$\Psi_{\boldsymbol{\alpha}} \equiv W_{T+\tau}(\boldsymbol{\alpha}) - \gamma(\boldsymbol{\alpha}) W_{T+\tau}(\boldsymbol{\beta}). \tag{5.4}$$

The function γ is a normalization factor such that at the time the investment decision is made the reference portfolio and the allocation have the same value:

$$\gamma(\boldsymbol{\alpha}) \equiv \frac{w_T(\boldsymbol{\alpha})}{w_T(\boldsymbol{\beta})}. \tag{5.5}$$

In this case the explicit expression of the objective in terms of the allocation $\boldsymbol{\alpha}$ reads:

$$\Psi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}' \mathbf{K} \mathbf{P}_{T+\tau}. \tag{5.6}$$

The constant matrix $\mathbf{K}$ in this expression is defined as follows:

$$\mathbf{K} \equiv \mathbf{I}_N - \frac{\mathbf{p}_T \boldsymbol{\beta}'}{\boldsymbol{\beta}' \mathbf{p}_T}, \tag{5.7}$$

where $\mathbf{I}_N$ is the identity matrix.

For example, mutual fund managers are evaluated every year against a benchmark that defines the fund's style. Therefore for mutual fund managers the horizon is one year and the objective is relative wealth with respect to the benchmark fund.

- **Net profits**

According to *prospect theory* some investors are more concerned with changes in wealth than with the absolute value of wealth, see Kahneman and Tversky (1979). Therefore the objective becomes:

$$\Psi_{\boldsymbol{\alpha}} \equiv W_{T+\tau}(\boldsymbol{\alpha}) - w_T(\boldsymbol{\alpha}). \tag{5.8}$$

The explicit expression of the objective in terms of the allocation reads in this case:

$$\Psi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}' (\mathbf{P}_{T+\tau} - \mathbf{p}_T). \tag{5.9}$$

For example, traders focus on their daily *profit and loss* (*P&L*). Therefore for a trader the investment horizon is one day and the net profits are his objective.

Notice that, in all its specifications, the objective is a linear function of the allocation and of a *market vector*:

$$\Psi_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}' \mathbf{M}. \tag{5.10}$$

The market vector $\mathbf{M}$ is a simple invertible affine transformation of the market prices at the investment horizon:

$$\mathbf{M} \equiv \mathbf{a} + \mathbf{B} \mathbf{P}_{T+\tau}, \tag{5.11}$$

where $\mathbf{a}$ is a suitable conformable vector and $\mathbf{B}$ is a suitable conformable invertible matrix. Indeed, from (5.3) the market vector for the absolute wealth objective follows from the choice:

$$\mathbf{a} \equiv \mathbf{0}, \quad \mathbf{B} \equiv \mathbf{I}_N; \tag{5.12}$$

from (5.6) the market vector for the relative wealth objective follows from the choice:

$$\mathbf{a} \equiv \mathbf{0}, \quad \mathbf{B} \equiv \mathbf{K}, \tag{5.13}$$

where $\mathbf{K}$ is defined in (5.7); from (5.9) the market vector for the net profits objective follows from the choice:

$$\mathbf{a} \equiv -\mathbf{p}_T, \quad \mathbf{B} \equiv \mathbf{I}_N. \tag{5.14}$$

The distribution of $\mathbf{M}$ can be easily computed from the distribution of the security prices $\mathbf{P}_{T+\tau}$ at the investment horizon and viceversa, see Appendix www.2.4. For instance, in terms of the characteristic function we obtain:

$$\phi_{\mathbf{M}}(\boldsymbol{\omega}) = e^{i\boldsymbol{\omega}'\mathbf{a}} \phi_{\mathbf{P}}(\mathbf{B}'\boldsymbol{\omega}). \tag{5.15}$$

Therefore, with a slight abuse of terminology, we refer to both $\mathbf{M}$ and $\mathbf{P}_{T+\tau}$ as the "market vector" or simply the "market".

From (5.10) it follows that the objective as a function of the allocation is *homogeneous of first degree*:

$$\Psi_{\lambda\boldsymbol{\alpha}} = \lambda\Psi_{\boldsymbol{\alpha}}; \tag{5.16}$$

and *additive*:

$$\Psi_{\boldsymbol{\alpha}+\boldsymbol{\beta}} = \Psi_{\boldsymbol{\alpha}} + \Psi_{\boldsymbol{\beta}}. \tag{5.17}$$

These properties allow to build and compare objectives that refer to complex portfolios of securities.

If the markets were deterministic, the investor could compute the objective relative to a given allocation as a deterministic function of that allocation, and thus he would choose the allocation that gives rise to the largest value of the objective.

For example, assume that the investor's objective is final wealth, i.e. (5.3). Suppose that the market prices grew linearly:

$$\mathbf{P}_{T+t} = \operatorname{diag}(\mathbf{p}_T)\,\mathbf{h}t, \tag{5.18}$$

where $\mathbf{h}$ is a constant vector. Then from (5.12) the market vector would read:

$$\mathbf{M} \equiv \mathbf{P}_{T+\tau} = \operatorname{diag}(\mathbf{p}_T)\,\mathbf{h}\tau. \tag{5.19}$$

Consequently, the investor would allocate all his money in the asset that performs the best over the investment horizon, which corresponds to the largest entry in the vector $\mathbf{h}$.

Instead, the market prices at the investment horizon are stochastic and therefore the market vector is a random variable, and so is the investor's objective.

For example, consider normally distributed market prices:

$$\mathbf{P}_{T+\tau} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{5.20}$$

If the investor focuses on final wealth, from (5.12) the market vector reads:

$$\mathbf{M} \equiv \mathbf{P}_{T+\tau}. \tag{5.21}$$

Thus the objective (5.10) is normally distributed:

$$\Psi_{\boldsymbol{\alpha}} \sim \mathrm{N}\left(\mu_{\boldsymbol{\alpha}}, \sigma_{\boldsymbol{\alpha}}^2\right), \tag{5.22}$$

where

$$\mu_{\boldsymbol{\alpha}} \equiv \boldsymbol{\mu}'\boldsymbol{\alpha}, \qquad \sigma_{\boldsymbol{\alpha}}^2 \equiv \boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}. \tag{5.23}$$

Since the objective is a random variable we need some tools to figure out in which sense a random variable is "larger" or is "better" than another one. We devote the rest of this chapter to this purpose.

We conclude this section remarking that the computation of the exact distribution of the objective $\Psi_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'\mathbf{M}$ is in general a formidable task. Indeed, the distribution of the market is easily obtained *once* the distribution of the prices is known, see (5.15). Nevertheless, the distribution of the prices is very hard to compute in general. Here we mention the *gamma approximation* of the investor's objective, a quite general approximate solution which has found a wide range of applications.

Consider the generic second-order approximation (3.108) for the prices of the securities in terms of the underlying market invariants $\mathbf{X}$, which we report here:

$$P_{T+\tau}^{(n)} \approx g^{(n)}(\mathbf{0}) + \mathbf{X}' \left.\frac{\partial g^{(n)}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{0}} + \frac{1}{2}\mathbf{X}' \left.\frac{\partial^2 g^{(n)}}{\partial \mathbf{x}\partial \mathbf{x}'}\right|_{\mathbf{x}=\mathbf{0}} \mathbf{X}, \tag{5.24}$$

where $n = 1, \ldots, N$. As we show in Appendix www.5.1, the investor's objective can be approximated by a quadratic function of the invariants:

$$\Psi_{\boldsymbol{\alpha}} \approx \Xi_{\boldsymbol{\alpha}} \equiv \theta_{\boldsymbol{\alpha}} + \boldsymbol{\Delta}_{\boldsymbol{\alpha}}'\mathbf{X} + \frac{1}{2}\mathbf{X}'\boldsymbol{\Gamma}_{\boldsymbol{\alpha}}\mathbf{X}, \tag{5.25}$$

where

$$\theta_{\boldsymbol{\alpha}} \equiv \sum_{n=1}^{N} \alpha_n a_n + \sum_{n,m=1}^{N} \alpha_n B_{nm} g^{(m)}(\mathbf{0}) \tag{5.26}$$

$$\boldsymbol{\Delta}_{\boldsymbol{\alpha}} \equiv \sum_{n,m=1}^{N} \alpha_n B_{nm} \left.\frac{\partial g^{(m)}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{0}} \tag{5.27}$$

$$\boldsymbol{\Gamma}_{\boldsymbol{\alpha}} \equiv \sum_{n,m=1}^{N} \alpha_n B_{nm} \left.\frac{\partial^2 g^{(m)}}{\partial \mathbf{x}\partial \mathbf{x}'}\right|_{\mathbf{x}=\mathbf{0}}; \tag{5.28}$$

and $\mathbf{a}$ and $\mathbf{B}$ are the coefficients (5.11) that determine the market.

In general the market invariants are sufficiently symmetric to be modeled appropriately by symmetrical distributions, such as elliptical or symmetric stable distributions, see (3.22), or (3.37), or (3.55), and comments thereafter.

Under this hypothesis it is possible to compute the distribution of the approximate objective (5.25) as represented by its characteristic function. In particular, assume that the invariants are normally distributed:

$$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{5.29}$$

Then we prove in Appendix www.5.1 that the characteristic function of the approximate objective (5.25) reads:

$$\begin{aligned}\phi_{\Xi_{\alpha}}(\omega) = \left|\mathbf{I}_K - i\omega\boldsymbol{\Gamma}_{\boldsymbol{\alpha}}\boldsymbol{\Sigma}\right|^{-\frac{1}{2}} & e^{i\omega\left(\theta_{\alpha}+\boldsymbol{\Delta}_{\alpha}'\boldsymbol{\mu}+\frac{1}{2}\boldsymbol{\mu}'\boldsymbol{\Gamma}_{\alpha}\boldsymbol{\mu}\right)} \\ & e^{-\frac{1}{2}[\boldsymbol{\Delta}_{\alpha}+\boldsymbol{\Gamma}_{\alpha}\boldsymbol{\mu}]'\boldsymbol{\Sigma}(\mathbf{I}_K-i\omega\boldsymbol{\Gamma}_{\alpha}\boldsymbol{\Sigma})^{-1}[\boldsymbol{\Delta}_{\alpha}+\boldsymbol{\Gamma}_{\alpha}\boldsymbol{\mu}]},\end{aligned} \tag{5.30}$$

where the explicit dependence on the allocation $\boldsymbol{\alpha}$ is easily recovered from (5.26)-(5.28).

5.2 Stochastic dominance

In this section we present the stochastic dominance approach to assess the distribution of the investor's objective. For further references, see Ingersoll (1987), Levy (1998) and Yamai and Yoshiba (2002).

Suppose that the investor can choose between an allocation $\boldsymbol{\alpha}$ that gives rise to the objective $\Psi_{\boldsymbol{\alpha}}$ and an allocation $\boldsymbol{\beta}$ that gives rise to the objective $\Psi_{\boldsymbol{\beta}}$. All the information necessary to make a decision as to which allocation is more advantageous is contained in the joint distribution of $\Psi_{\boldsymbol{\alpha}}$ and $\Psi_{\boldsymbol{\beta}}$.

When confronted with two different objectives $\Psi_{\boldsymbol{\alpha}}$ and $\Psi_{\boldsymbol{\beta}}$, it is natural to first check whether in all possible scenarios one objective is larger than the other, see the left plot in Figure 5.1. When this happens, the objective $\Psi_{\boldsymbol{\alpha}}$, or the allocation $\boldsymbol{\alpha}$, is said to *strongly dominate* the objective $\Psi_{\boldsymbol{\beta}}$, or the allocation $\boldsymbol{\beta}$:

$$\text{strong dom.: } \Psi_{\boldsymbol{\alpha}} \geq \Psi_{\boldsymbol{\beta}} \text{ in all scenarios.} \tag{5.31}$$

In other words, strong dominance arises when the difference of the objectives relative to two allocations is a positive random variable. Therefore, an equivalent definition of strong dominance reads as follows in terms of the cumulative distribution function of the difference of the objectives:

$$\text{strong dom.: } F_{\Psi_{\alpha}-\Psi_{\beta}}(0) \equiv \mathbb{P}\{\Psi_{\boldsymbol{\alpha}} - \Psi_{\boldsymbol{\beta}} \leq 0\} = 0. \tag{5.32}$$

We call strong dominance also *order zero dominance*, for reasons that will become clear below.

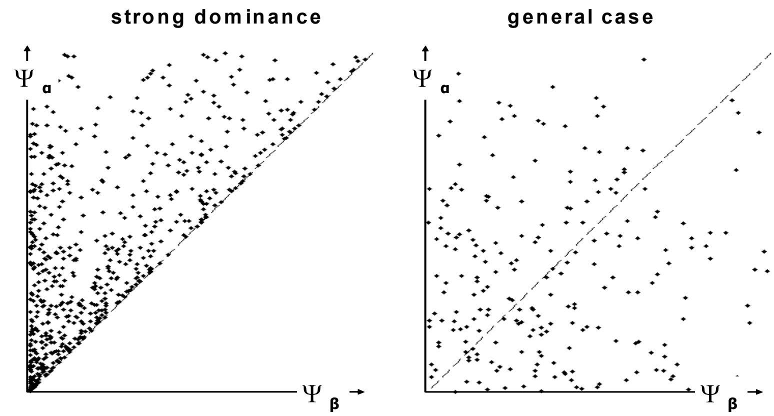

Fig. 5.1. Strong dominance

For example, suppose that the objective relative to one allocation has a chi-square distribution with two degrees of freedom:

$$\Psi_{\boldsymbol{\alpha}} \sim \chi_2^2; \tag{5.33}$$

and that the objective relative to another allocation has a chi-square distribution with one degree of freedom:

$$\Psi_{\boldsymbol{\beta}} \sim \chi_1^2. \tag{5.34}$$

Assume that $\Psi_{\boldsymbol{\alpha}} = \Psi_{\boldsymbol{\beta}} + Y$ where $Y \sim \chi_1^2$ is independent of $\Psi_{\boldsymbol{\beta}}$. Then $\Psi_{\boldsymbol{\alpha}}$ strongly dominates $\Psi_{\boldsymbol{\beta}}$. With this example we generated the plot on the left in Figure 5.1.

Nevertheless, strong dominance cannot be a general criterion to evaluate allocations.

In the first place, strong dominance never takes place, for if it did, *arbitrage* opportunities, i.e. "free lunches" would arise. Instead, in general an allocation $\boldsymbol{\alpha}$ gives rise to an objective $\Psi_{\boldsymbol{\alpha}}$ that in some scenarios is larger and in some scenarios is smaller than the objective $\Psi_{\boldsymbol{\beta}}$ stemming from a different allocation $\boldsymbol{\beta}$, see the plot on the right hand side in Figure 5.1.

Secondly, the definition of strong dominance relies on the joint distribution of the two objectives $\Psi_{\boldsymbol{\alpha}}$ and $\Psi_{\boldsymbol{\beta}}$, which is necessary to compute the distribution of their difference (5.32). Nevertheless, the two allocations are mutually exclusive, i.e. the investor either chooses one or the other. Therefore the interplay of the two allocations should not have an effect on the decision and thus any criterion to rank allocations should focus on comparing the marginal distributions of the objectives.

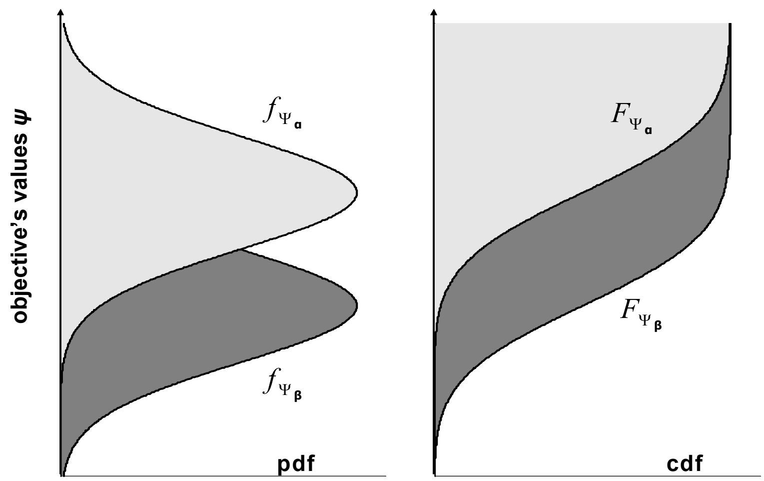

Fig. 5.2. Weak dominance

A different approach that compares two allocations only in terms of their marginal distributions is inspired from a plot of the possible values of the objectives on the vertical axis and the respective probability density functions on the horizontal axis as in Figure 5.2. We would be prone to choose an allocation $\boldsymbol{\alpha}$ over another allocation $\boldsymbol{\beta}$ if the probability density function of the ensuing objective were concentrated around larger values than for the other allocation. This condition is expressed more easily in terms of the cumulative distribution function. The objective $\Psi_{\boldsymbol{\alpha}}$, or the allocation $\boldsymbol{\alpha}$, is said to *weakly dominate* the objective $\Psi_{\boldsymbol{\beta}}$, or the allocation $\boldsymbol{\beta}$, if the following condition holds true (notice the "wrong" direction of the inequality):

$$\text{weak dom.: } F_{\Psi_\alpha}(\psi) \leq F_{\Psi_\beta}(\psi) \text{ for all } \psi \in (-\infty, +\infty) . \tag{5.35}$$

Weak dominance is also called *first-order dominance*, for reasons to become clear below.

Comparing Figure 5.2 with Figure 1.2 we obtain a more intuitive equivalent expression for weak dominance in terms of the inverse of the cumulative distribution function, namely the quantile. The objective $\Psi_{\boldsymbol{\alpha}}$, or the allocation $\boldsymbol{\alpha}$, is said to weakly dominate the objective $\Psi_{\boldsymbol{\beta}}$, or the allocation $\boldsymbol{\beta}$, if the following condition holds true:

$$\text{weak dom: } Q_{\Psi_\alpha}(p) \geq Q_{\Psi_\beta}(p) \text{ for all } p \in (0, 1) . \tag{5.36}$$

This representation is more intuitive than (5.35), due to the "correct" direction of the inequality.

Weak dominance is not as restrictive a condition as strong dominance: if an allocation strongly dominates another one it also weakly dominates it, but the opposite is not true.

For example in Figure 5.2 we consider the case:

$$\Psi_{\boldsymbol{\alpha}} \sim \mathrm{N}(1,1), \quad \Psi_{\boldsymbol{\beta}} \sim \mathrm{N}(0,1). \tag{5.37}$$

From the expression of the normal cumulative distribution function (1.68) we obtain:

$$\begin{aligned} F_{\Psi_{\alpha}}(\psi) &\equiv \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{\psi-1}{\sqrt{2}}\right)\right] \\ &\leq \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{\psi}{\sqrt{2}}\right)\right] \equiv F_{\Psi_{\beta}}(\psi). \end{aligned} \tag{5.38}$$

Therefore, the allocation $\boldsymbol{\alpha}$ weakly dominates $\boldsymbol{\beta}$.

Assume that the two objectives are independent. Then

$$\Psi_{\boldsymbol{\alpha}} - \Psi_{\boldsymbol{\beta}} \sim \mathrm{N}(1,2), \tag{5.39}$$

or equivalently

$$F_{\Psi_{\alpha}-\Psi_{\beta}}(0) = \frac{1}{2}\left[1+\operatorname{erf}\left(-\frac{1}{2}\right)\right] > 0. \tag{5.40}$$

Therefore, from (5.32) the allocation $\boldsymbol{\alpha}$ does not strongly dominate the allocation $\boldsymbol{\beta}$.

A third way to express weak dominance is the following. Suppose that v is the realization of a random variable V which spans the unit interval, such as the standard uniform distribution:

$$V \sim \mathrm{U}([0,1]). \tag{5.41}$$

Applying the cumulative distribution function $F_{\Psi_{\beta}}$ to both sides of the inequality in (5.36) we obtain:

$$\text{weak dom.: } F_{\Psi_{\beta}}\left(Q_{\Psi_{\alpha}}(V)\right) \geq V \text{ in all scenarios.} \tag{5.42}$$

Comparing this expression with the definition of strong dominance (5.31) we can say that the objective $\Psi_{\boldsymbol{\alpha}}$, or the allocation $\boldsymbol{\alpha}$, weakly dominates the objective $\Psi_{\boldsymbol{\beta}}$, or the allocation $\boldsymbol{\beta}$, if the distribution on the left hand side in (5.42) strongly dominates the uniform distribution on the right hand side, see Figure 5.1.

In particular, from (2.26) the grade of the objective is uniformly distributed on the unit interval: $F_{\Psi_{\alpha}}(\Psi_{\boldsymbol{\alpha}}) \stackrel{d}{=} V$; and from (2.27) the quantile of the

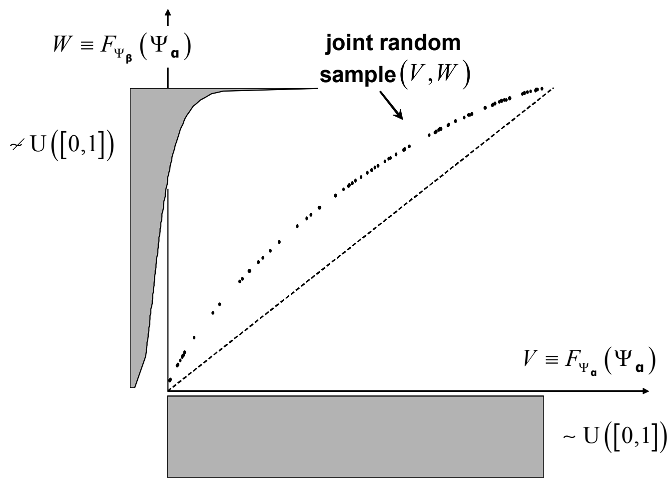

Fig. 5.3. Weak dominance in terms of strong dominance

grade is distributed as the objective: $\Psi_{\boldsymbol{\alpha}} \stackrel{d}{=} Q_{\Psi_{\alpha}}(V)$. Therefore from (5.42) an allocation $\boldsymbol{\alpha}$ weakly dominates an allocation $\boldsymbol{\beta}$ if $F_{\Psi_{\beta}}(\Psi_{\boldsymbol{\alpha}})$ strongly dominates $F_{\Psi_{\alpha}}(\Psi_{\boldsymbol{\alpha}})$.

Figure 5.3 refers to our example (5.37), where the allocation $\boldsymbol{\alpha}$ weakly dominates the allocation $\boldsymbol{\beta}$. Indeed, as in Figure 5.1, all the joint outcomes of $F_{\Psi_{\alpha}}(\Psi_{\boldsymbol{\alpha}})$ and $F_{\Psi_{\beta}}(\Psi_{\boldsymbol{\alpha}})$ lie above the diagonal and thus $F_{\Psi_{\beta}}(\Psi_{\boldsymbol{\alpha}})$ strongly dominates $F_{\Psi_{\alpha}}(\Psi_{\boldsymbol{\alpha}})$.

Although weak, or first-order, dominance is not a criterion as restrictive as strong dominance, even first-order dominance hardly ever occurs. To cope with this problem we need to introduce even weaker types of dominance, such as *second-order stochastic dominance* (SSD).

The rationale behind second-order stochastic dominance is the following: we would be prone to choose the distribution $\Psi_{\boldsymbol{\alpha}}$ over the distribution $\Psi_{\boldsymbol{\beta}}$ if, for any given benchmark level ψ of the objective, the events where $\Psi_{\boldsymbol{\alpha}}$ underperforms the benchmark level are less harmful than for $\Psi_{\boldsymbol{\beta}}$. In formulas, for all $\psi \in (-\infty, +\infty)$ the following inequality must hold:

$$\text{SSD: } \mathrm{E}\left\{(\Psi_{\boldsymbol{\alpha}} - \psi)^{-}\right\} \geq \mathrm{E}\left\{(\Psi_{\boldsymbol{\beta}} - \psi)^{-}\right\}, \tag{5.43}$$

where the "minus" denotes the negative part. If (5.43) holds true, $\Psi_{\boldsymbol{\alpha}}$ is said to second-order dominate $\Psi_{\boldsymbol{\beta}}$.

An equivalent formulation of second-order stochastic dominance is the following, see Ingersoll (1987) for a proof. The objective $\Psi_{\boldsymbol{\alpha}}$, or the allocation

$\boldsymbol{\alpha}$, second-order dominates the objective $\Psi_{\boldsymbol{\beta}}$, or the allocation $\boldsymbol{\beta}$, if for all $\psi \in (-\infty, +\infty)$ the following inequality holds:

$$\text{SSD: } \mathcal{I}^2 \left[f_{\Psi_{\alpha}} \right] (\psi) \leq \mathcal{I}^2 \left[f_{\Psi_{\beta}} \right] (\psi) , \tag{5.44}$$

where $\mathcal{I}^2$ the iterated integral $(B.27)$ of the pdf:

$$\mathcal{I}^2 \left[f_{\Psi} \right] (\psi) \equiv \mathcal{I} \left[F_{\Psi} \right] (\psi) \equiv \int_{-\infty}^{\psi} F_{\Psi} (s) \, ds. \tag{5.45}$$

Second-order stochastic dominance is a less restrictive condition than weak dominance. Indeed, applying the integration operator $\mathcal{I}$ on both sides of (5.35) we see that first-order dominance implies second-order dominance, although the opposite is not true in general.

If even second-order dominance does not take place, we must pursue weaker and weaker criteria. More in general, we say that the objective $\Psi_{\boldsymbol{\alpha}}$, or the allocation $\boldsymbol{\alpha}$, *order-q dominates* the objective $\Psi_{\boldsymbol{\beta}}$, or the allocation $\boldsymbol{\beta}$, if for all $\psi \in (-\infty, +\infty)$ the following inequality holds:

$$q\text{-dom.: } \mathcal{I}^q \left[f_{\Psi_{\alpha}} \right] (\psi) \leq \mathcal{I}^q \left[f_{\Psi_{\beta}} \right] (\psi) . \tag{5.46}$$

Notice that first-order dominance (5.35) and second-order dominance (5.44) are particular cases of (5.46).

Applying the integration operator to both sides of (5.46) we see that order q dominance implies order $(q+1)$ dominance, although the opposite is not true in general.

Recalling that we renamed strong dominance as zero-order dominance, we can write all the above implications in compact form as follows:

$$0\text{-dom.} \Rightarrow 1\text{-dom.} \Rightarrow \cdots \Rightarrow q\text{-dom.} \tag{5.47}$$

Therefore in theory we only need to check that one allocation dominates another for a certain degree, as dominance for higher degrees follows. In practice the stochastic dominance approach to evaluating allocations displays major drawbacks.

First of all, the intuitive meaning behind dominance of orders higher than two is not evident.

Secondly, the computation of the generic q-th cumulative distribution is not practically feasible in most situations.

Finally, but most importantly, there is no guarantee that there exists an order q such that a portfolio stochastically dominates or is dominated by another: consequently, the investor might not be able to rank his potential investments and thus choose an allocation. Intuitively, this happens because the objective is stochastic: a deterministic variable can be represented by a point on the real line, whereas a random variable is represented by a function, such as the cumulative distribution function. Functions are infinite-dimensional vectors, i.e. points in an infinite-dimensional space, see Appendix B. In dimensions higher than one there exists no natural way to order points.

5.3 Satisfaction

The main drawback of the dominance approach to ranking two allocations $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is that two generic allocations might not be comparable, in the sense that neither of the respective objectives dominates the other.

To solve this problem, we summarize all the features of a given allocation $\boldsymbol{\alpha}$ into one single number $\mathcal{S}$ that indicates the respective degree of satisfaction:

$$\boldsymbol{\alpha} \mapsto \mathcal{S}(\boldsymbol{\alpha}) . \tag{5.48}$$

The investor will then choose the allocation that corresponds to the highest degree of satisfaction.

For example, the expected value of the investor's objective is a number that depends on the allocation:

$$\boldsymbol{\alpha} \mapsto \mathcal{S}(\boldsymbol{\alpha}) \equiv \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} . \tag{5.49}$$

As such, it is an index of satisfaction.

Since there exists no univocal way to summarize all the information contained in an allocation into one number, we discuss here potential features that an index of satisfaction may display, see also Frittelli and Rosazza Gianin (2002).

- **Money-equivalence**

An index of satisfaction is *money-equivalent* if it is naturally measured in units of money. This is a desirable feature, as money is "the" measure in finance.

Furthermore, money-equivalence is necessary for consistence. Indeed, consider an investor with a given objective such as absolute wealth as in (5.3) or relative wealth, as in (5.4), or net profits, as in (5.8), or possibly other specifications. In all the specifications the objective is measured in terms of money. Since in a deterministic environment the most natural index of satisfaction is the objective, it is intuitive to require that a generic index of satisfaction be measured in the same units as the objective.

For example, the expected value of the objective (5.49) has the same dimension as the objective, which is money, and thus it is a money-equivalent index of satisfaction.

The concept of money-equivalence contrasts that of *scale-invariance*, or *homogeneity of degree zero*. Scale invariant indices of satisfaction are dimensionless measures that satisfy the following relation:

$$\mathcal{S}(\lambda\boldsymbol{\alpha}) = \mathcal{S}(\boldsymbol{\alpha}) , \text{ for all } \lambda > 0. \tag{5.50}$$

Scale invariant indices of satisfaction provide a tool to normalize and evaluate portfolios that differ in size.

Although in the sequel we will be concerned mainly with money-equivalent indices of satisfaction, we present the most notable scale invariant index of satisfaction, namely the *Sharpe ratio*, which is defined as follows:

$$\mathrm{SR}(\boldsymbol{\alpha}) \equiv \frac{\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\}}{\mathrm{Sd}\{\Psi_{\boldsymbol{\alpha}}\}}. \tag{5.51}$$

The rationale behind the Sharpe ratio is the following: a high standard deviation is a drawback if the expected value of the objective is positive, because it adds uncertainty to a potentially satisfactory outcome.

- **Estimability**

An index of satisfaction is *estimable* if the satisfaction associated with a generic allocation $\boldsymbol{\alpha}$ is fully determined by the marginal distribution of the investor's objective $\Psi_{\boldsymbol{\alpha}}$, which can be absolute wealth, relative wealth, net gains, etc. In other words, two allocations that give rise to two objectives with the same distribution are fully equivalent for the investor.

For example, an allocation of a thousand dollars in cash and a thousand dollars in a stock of a company quoted on the NYSE is considered fully equivalent to an investment of a thousand dollars in a currency pegged to the dollar and a thousand dollars in the same stock as quoted on the DAX.

In other words, when an index of the satisfaction is estimable, the satisfaction associated with the allocation $\boldsymbol{\alpha}$ is a functional of any of the equivalent representations of the distribution of the objective $\Psi_{\boldsymbol{\alpha}}$, namely the probability density function $f_{\Psi_{\boldsymbol{\alpha}}}$, the cumulative distribution function $F_{\Psi_{\boldsymbol{\alpha}}}$, or the characteristic function $\phi_{\Psi_{\boldsymbol{\alpha}}}$. Therefore, in order to be estimable, the simple map (5.48) must expand into the following chain of maps:

$$\boldsymbol{\alpha} \mapsto \Psi_{\boldsymbol{\alpha}} \mapsto \left(f_{\Psi_{\boldsymbol{\alpha}}}, F_{\Psi_{\boldsymbol{\alpha}}}, \phi_{\Psi_{\boldsymbol{\alpha}}}\right) \mapsto \mathcal{S}(\boldsymbol{\alpha}). \tag{5.52}$$

The concept of estimability is known in the financial literature also under the name of *law invariance*, see Kusuoka (2001).

For example, the expected value is a functional of the probability density function of the objective:

$$f_{\psi} \mapsto \mathrm{E}\{\Psi\} \equiv \int_{\mathbb{R}} \psi f_{\psi}(\psi)\, d\psi. \tag{5.53}$$

Therefore, the expected value is an estimable index of satisfaction:

$$\boldsymbol{\alpha} \mapsto \Psi_{\boldsymbol{\alpha}} \mapsto f_{\Psi_{\boldsymbol{\alpha}}} \mapsto \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\}. \tag{5.54}$$

- **Sensibility**

Due to the non-satiation principle underlying the investor's objective, if an objective $\Psi_{\boldsymbol{\alpha}}$ is larger than an objective $\Psi_{\boldsymbol{\beta}}$ in all scenarios, then the satisfaction that the investors derive from $\Psi_{\boldsymbol{\alpha}}$ should be greater than the satisfaction they derive from $\Psi_{\boldsymbol{\beta}}$:

$$\Psi_{\boldsymbol{\alpha}} \geq \Psi_{\boldsymbol{\beta}} \text{ in all scenarios } \Rightarrow \mathcal{S}(\boldsymbol{\alpha}) \geq \mathcal{S}(\boldsymbol{\beta}). \tag{5.55}$$

We call this feature *sensibility* because it is the minimum requirement that any index of satisfaction needs to verify. Sensibility is also called *monotonicity* in the financial literature, see Artzner, Delbaen, Eber, and Heath (1999).

For example, the expected value of the objective (5.49) is a sensible index of satisfaction, since it trivially satisfies:

$$\Psi_{\boldsymbol{\alpha}} \geq \Psi_{\boldsymbol{\beta}} \text{ in all scenarios } \Rightarrow \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} \geq \mathrm{E}\{\Psi_{\boldsymbol{\beta}}\}. \tag{5.56}$$

By comparing the sensibility condition (5.55) with (5.31) we notice that in order for the index of satisfaction to be sensible it must be consistent with strong dominance. In other words, if an allocation $\boldsymbol{\alpha}$ happens to strongly dominate an allocation $\boldsymbol{\beta}$, any sensible criterion should prefer the former to the latter.

Although we cannot rely on strong dominance as a criterion to compare allocations, we should always make sure that any possible criterion is consistent with strong dominance.

- **Consistence with stochastic dominance**

Sensibility stems from the intuitive non-satiation argument that the larger in a strong sense the investor's objective, the more satisfied the investor. Similarly, we can apply the non-satiation argument to weaker concepts of dominance.

For instance, if the marginal distribution of the objective $\Psi_{\boldsymbol{\alpha}}$ of an allocation $\boldsymbol{\alpha}$ is shifted upward with respect to the marginal distribution of the objective $\Psi_{\boldsymbol{\beta}}$ of an allocation $\boldsymbol{\beta}$ as in Figure 5.2, then the satisfaction from $\Psi_{\boldsymbol{\alpha}}$ should be greater than the satisfaction from $\Psi_{\boldsymbol{\beta}}$. In formulas:

$$Q_{\Psi_{\boldsymbol{\alpha}}}(p) \geq Q_{\Psi_{\boldsymbol{\beta}}}(p) \text{ for all } p \in (0,1) \Rightarrow \mathcal{S}(\boldsymbol{\alpha}) \geq \mathcal{S}(\boldsymbol{\beta}). \tag{5.57}$$

By comparing this expression with (5.36) we realize that an index of satisfaction for which the above relation holds is consistent with weak dominance.

For example, the expected value of the objective (5.49) is consistent with weak dominance. Indeed with a change of variable we can verify the following equality:

$$\mathrm{E}\{\Psi\} \equiv \int_{-\infty}^{+\infty} \psi f_{\psi}(\psi)\, d\psi = \int_{0}^{1} Q_{\Psi}(u)\, du. \tag{5.58}$$

Therefore (5.57) is satisfied.

Consistence with weak dominance is a stronger requirement on the index of satisfaction than consistence with strong dominance: if and index of satisfaction is consistent with weak dominance it is sensible, i.e. it is consistent with strong dominance, but the opposite is not true in general.

Nevertheless, in the case of estimable indices of satisfaction the two statements are equivalent. Indeed, in Appendix www.5.2 we follow a personal communication by D. Tasche to prove:

$$\mathcal{S} \text{ estimable} + \mathcal{S} \text{ sensible } \Rightarrow \mathcal{S} \text{ weak dom. consistent.} \tag{5.59}$$

In general, an index of satisfaction $\mathcal{S}$ is consistent with q-th order dominance if, whenever an allocation $\boldsymbol{\alpha}$ dominates an allocation $\boldsymbol{\beta}$ at order q, then the satisfaction from $\boldsymbol{\alpha}$ is larger than the satisfaction from $\boldsymbol{\beta}$:

$$\Psi_{\boldsymbol{\alpha}} \ q\text{-dom. } \Psi_{\boldsymbol{\beta}} \Rightarrow \mathcal{S}(\boldsymbol{\alpha}) \geq \mathcal{S}(\boldsymbol{\beta}). \tag{5.60}$$

In particular, sensibility corresponds to consistency with zero-order dominance, see (5.55); and consistency with weak dominance corresponds to consistency with first-order dominance, see (5.57).

Given the sequence of implications (5.47) on the degrees of dominance, the reverse sequence holds for the consistency of an index of satisfaction with the degree of dominance:

$$\begin{aligned} q\text{-dom. consistence} \Rightarrow \cdots \Rightarrow\ & 1\text{-dom. consistence} \\ \Rightarrow\ & 0\text{-dom. consistence.} \end{aligned} \tag{5.61}$$

For example, the expected value of the objective (5.49) is consistent with second-order dominance. We prove this result in a broader context in Section 5.4. Therefore, it is consistent with weak dominance (first-order dominance), and therefore it is sensible, i.e. it is consistent with order-zero dominance.

- **Constancy**

If the markets were deterministic, the non-satiation principle would imply that the investor's objective, no matter whether it is absolute wealth, or relative wealth, or net profits, would serve as a suitable index of satisfaction, see p. 241.

Therefore, if there exists an allocation $\mathbf{b}$ that yields a deterministic objective $\psi_{\mathbf{b}}$, it is reasonable to require that the index coincide with the objective:

$$\Psi_{\mathbf{b}} \equiv \psi_{\mathbf{b}} \Rightarrow \mathcal{S}(\mathbf{b}) = \psi_{\mathbf{b}}. \tag{5.62}$$

This feature is called *constancy*.

For example, the expected value of the objective (5.49) is a constant index of satisfaction:

$$\Psi_{\mathbf{b}} \equiv \psi_{\mathbf{b}} \Rightarrow \mathrm{E}\{\Psi_{\mathbf{b}}\} = \psi_{\mathbf{b}}. \tag{5.63}$$

As an example of a deterministic objective, consider the case where the objective is absolute wealth and the allocation $\mathbf{b}$ is an investment in zero-coupon bonds that expire at the investment horizon.

- **Positive homogeneity**

The investor's objective is positive homogeneous of degree one, see (5.16). In other words, if we rescale the allocation by a given positive factor the objective is rescaled by the same factor:

$$\Psi_{\lambda\boldsymbol{\alpha}} = \lambda\Psi_{\boldsymbol{\alpha}}, \quad \text{for all } \lambda \geq 0. \tag{5.64}$$

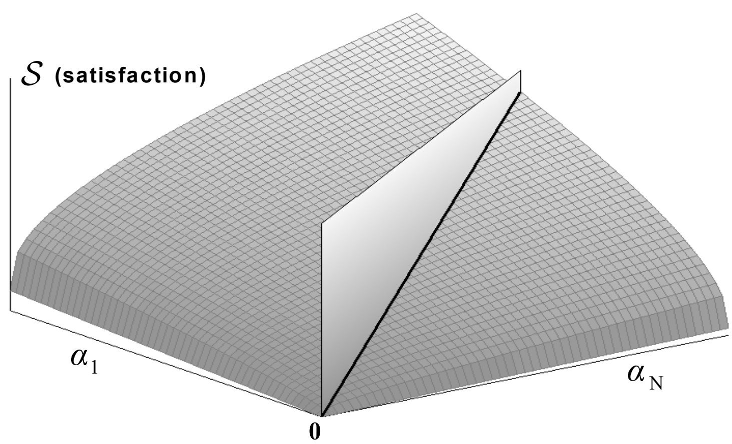

Fig. 5.4. Positive homogeneity of satisfaction index

It would be intuitive if an index of satisfaction shared the same property: loosely speaking, an index of satisfaction is homogeneous if doubling the investment makes the investor twice as happy. More precisely, a satisfaction index is *positive homogenous* (of degree one) if rescaling the allocation by a generic positive factor λ implies that satisfaction is rescaled by the same factor:

$$\mathcal{S}(\lambda\boldsymbol{\alpha}) = \lambda\mathcal{S}(\boldsymbol{\alpha}), \quad \text{for all } \lambda \geq 0. \tag{5.65}$$

It is easy to interpret positive homogeneity geometrically, see Figure 5.4. An index of satisfaction is positive homogenous if satisfaction grows linearly in any radial direction stemming from the origin of the allocation space.

For example, the expected value of the objective (5.49) is a positive homogeneous index of satisfaction. Indeed from (5.64) we obtain:

$$\mathrm{E}\{\Psi_{\lambda\boldsymbol{\alpha}}\} = \mathrm{E}\{\lambda\Psi_{\boldsymbol{\alpha}}\} = \lambda\,\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\}. \tag{5.66}$$

Positive homogeneous functions enjoy the following special property, first discovered by Euler:

$$\mathcal{S}(\boldsymbol{\alpha}) = \sum_{n=1}^{N} \alpha_n \frac{\partial \mathcal{S}(\boldsymbol{\alpha})}{\partial \alpha_n}. \tag{5.67}$$

In the case of positive homogeneous indices of satisfaction this property has a nice interpretation: satisfaction is the sum of the contributions from each security. The contribution to satisfaction from the generic n-th security in turn is the product of the amount α_n of that security times the marginal contribution to satisfaction $\partial\mathcal{S}/\partial\alpha_n$ which that security provides. Furthermore, the vector of marginal contributions to satisfaction is scale-invariant: if the allocation is multiplied by a positive factor, the marginal contribution $\partial\mathcal{S}/\partial\alpha_n$ does not change.

For example, if the objective is absolute wealth

$$\Psi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\mathbf{P}_{T+\tau}, \tag{5.68}$$

the Euler decomposition of the expected value yields:

$$\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} = \sum_{n=1}^{N} \alpha_n\,\mathrm{E}\left\{P_{T+\tau}^{(n)}\right\}. \tag{5.69}$$

In this case the contribution to satisfaction of the n-th security factors into the security's amount times the expected value of that security at the investment horizon.

- **Translation invariance**

The investor's objective is not only positive homogeneous, it is also additive, see (5.17): if we add two portfolios $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ the ensuing objective is the sum of the two separate objectives:

$$\Psi_{\boldsymbol{\alpha}+\boldsymbol{\beta}} = \Psi_{\boldsymbol{\alpha}} + \Psi_{\boldsymbol{\beta}}. \tag{5.70}$$

Since the objectives are random variables the satisfaction ensuing from the sum of two random variables can be completely unrelated to the satisfaction that the investor draws from the separate portfolios.

Nevertheless, consider an allocation $\mathbf{b}$ that yields a deterministic objective $\psi_{\mathbf{b}}$. In this case the distribution of the objective relative to the joint allocation

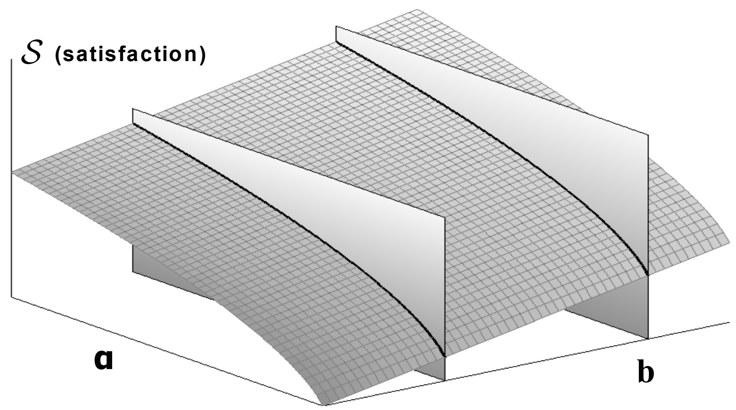

Fig. 5.5. Translation invariance of satisfaction index

$\boldsymbol{\alpha} + \mathbf{b}$ is simply a shift of the distribution of $\Psi_{\boldsymbol{\alpha}}$ by the fixed amount $\psi_{\mathbf{b}}$. If the index of satisfaction satisfies (5.62), this is the satisfaction provided by the allocation $\mathbf{b}$. Therefore, it would be intuitive if an index of satisfaction shifted by the same amount:

$$\mathcal{S}(\boldsymbol{\alpha} + \mathbf{b}) = \mathcal{S}(\boldsymbol{\alpha}) + \psi_{\mathbf{b}}. \tag{5.71}$$

This property is called *translation invariance*. It is easy to interpret translation invariance geometrically, see Figure 5.5.

Notice that translation invariance implies that the index of satisfaction is measured in terms of money. Without loss of generality, we can normalize the deterministic allocation to yield one unit of currency. Therefore we can restate the translation invariance property as follows:

$$\Psi_{\mathbf{b}} \equiv 1 \Rightarrow \mathcal{S}(\boldsymbol{\alpha} + \lambda \mathbf{b}) = \mathcal{S}(\boldsymbol{\alpha}) + \lambda. \tag{5.72}$$

This expression follows from (5.71) once we take into account the positive homogeneity of the objective (5.64).

For example, the expected value of the objective (5.49) is a translation-invariant index of satisfaction. Indeed, from (5.70) and (5.66) we obtain:

$$\Psi_{\mathbf{b}} \equiv 1 \Rightarrow \mathrm{E}\{\Psi_{\boldsymbol{\alpha}+\lambda\mathbf{b}}\} = \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} + \lambda. \tag{5.73}$$

- **Sub- and super- additivity**

The translation invariance property (5.71) can be interpreted as additivity in a deterministic environment. Indeed, the index of satisfaction of a deterministic allocation is the investor's objective, see (5.62). Therefore (5.71) reads:

$$\Psi_{\mathbf{b}} \equiv 1 \Rightarrow \mathcal{S}(\boldsymbol{\alpha}+\mathbf{b}) = \mathcal{S}(\boldsymbol{\alpha}) + \mathcal{S}(\mathbf{b}). \tag{5.74}$$

In the general non-deterministic case additivity is too strong a constraint: the satisfaction ensuing from two portfolios together could be larger or smaller than the sum of the separate levels of satisfaction. A measure of satisfaction is *super-additive*[1] if for any two allocations $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ the following inequality is satisfied:

$$\mathcal{S}(\boldsymbol{\alpha}+\boldsymbol{\beta}) \geq \mathcal{S}(\boldsymbol{\alpha}) + \mathcal{S}(\boldsymbol{\beta}). \tag{5.75}$$

The rationale behind this condition is that the interplay between the allocation $\boldsymbol{\alpha}$ and the allocation $\boldsymbol{\beta}$ provides a diversification effect that satisfies the investor more than the two allocations separately.

Similarly, a measure of satisfaction is *sub-additive* if for any two allocations $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$:

$$\mathcal{S}(\boldsymbol{\alpha}+\boldsymbol{\beta}) \leq \mathcal{S}(\boldsymbol{\alpha}) + \mathcal{S}(\boldsymbol{\beta}). \tag{5.76}$$

The rationale behind this condition is that the interplay between the allocation $\boldsymbol{\alpha}$ and the allocation $\boldsymbol{\beta}$ provides a diversification effect that the investor does not appreciate.

For example, the expected value of the objective (5.49) is an additive index of satisfaction:

$$\mathrm{E}\{\Psi_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\} = \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} + \mathrm{E}\{\Psi_{\boldsymbol{\beta}}\}. \tag{5.77}$$

This follows immediately from the additivity of the objective (5.70). Therefore the expected value is both sub- and super-additive.

- **Co-monotonic additivity**

Two allocations $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$ are *co-monotonic* if they give rise to co-monotonic objectives, i.e. such that one is a deterministic increasing function of the other, see (2.35).

For example, consider a market of two securities: a stock that trades at the price S_t and a call option on that stock with strike K that expires at the investment horizon. Consider the following two allocations:

$$\boldsymbol{\alpha} \equiv (1,0)', \quad \boldsymbol{\delta} \equiv (0,1)'. \tag{5.78}$$

[1] The financial literature that focuses on measures of risk rather than measures of satisfaction reverses the inequalities in the following expressions, see Artzner, Delbaen, Eber, and Heath (1999).

In words, the allocation $\boldsymbol{\alpha}$ is one share of the stock and the allocation $\boldsymbol{\delta}$ is a call option on one share of that stock. Consider an investor whose objective is final wealth. Then

$$\Psi_{\boldsymbol{\alpha}} = S_{T+\tau}, \quad \Psi_{\boldsymbol{\delta}} = \max\left(\Psi_{\boldsymbol{\alpha}} - K, 0\right). \tag{5.79}$$

The objective $\Psi_{\boldsymbol{\delta}}$ is an increasing function of the objective $\Psi_{\boldsymbol{\alpha}}$, once we regularize the call payoff as in (2.37). Thus the two allocations are co-monotonic.

A combination of co-monotonic allocations does not provide a genuine diversification effect: an extreme event in one of them is reflected in an extreme event in the other.

An index of satisfaction is *co-monotonic additive*, if it properly takes this phenomenon into account. From the remarks following (5.75), a co-monotonic additive index of satisfaction satisfies:

$$(\boldsymbol{\alpha}, \boldsymbol{\delta}) \text{ co-monotonic } \Rightarrow \mathcal{S}\left(\boldsymbol{\alpha} + \boldsymbol{\delta}\right) = \mathcal{S}\left(\boldsymbol{\alpha}\right) + \mathcal{S}\left(\boldsymbol{\delta}\right). \tag{5.80}$$

Loosely speaking, a co-monotonic additive index of satisfaction is "derivative-proof".

Since from (5.77) the expected value of the objective is an additive index of satisfaction, in particular it is co-monotonic additive.

- **Concavity/convexity**

We discussed above a few potential features of an index of satisfaction such as positive homogeneity, which refers to rescaling an allocation, and translation invariance, sub-additivity, super-additivity and co-monotonic additivity, which refer to summing allocations. These properties together help determining the level of satisfaction from a joint allocation $\lambda\boldsymbol{\alpha} + \mu\boldsymbol{\beta}$ in terms of the satisfaction $\mathcal{S}\left(\boldsymbol{\alpha}\right)$ and $\mathcal{S}\left(\boldsymbol{\beta}\right)$ from the separate portfolios $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and the respective amounts λ and μ of each portfolio.

In practical situations the investor is not interested in evaluating all the possible allocations spanned by two potential investments $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Instead, due to budget or liquidity constraints, investors typically focus on the satisfaction they draw from weighted averages of the two potential allocations, which include the two separate allocations as special cases.

An index of satisfaction is *concave* if for all $\lambda \in [0, 1]$ the following inequality holds:

$$\mathcal{S}\left(\lambda\boldsymbol{\alpha} + (1-\lambda)\boldsymbol{\beta}\right) \geq \lambda\mathcal{S}\left(\boldsymbol{\alpha}\right) + (1-\lambda)\,\mathcal{S}\left(\boldsymbol{\beta}\right). \tag{5.81}$$

Notice that concavity is implied by the joint assumptions of positive homogeneity (5.65) and super-additivity (5.75).

Similarly, an index of satisfaction is *convex* if for all $\lambda \in [0, 1]$ the following inequality holds:

$$\mathcal{S}\left(\lambda\boldsymbol{\alpha}+(1-\lambda)\boldsymbol{\beta}\right)\leq\lambda\mathcal{S}\left(\boldsymbol{\alpha}\right)+(1-\lambda)\mathcal{S}\left(\boldsymbol{\beta}\right). \tag{5.82}$$

It is immediate to verify that convexity is implied by the joint assumptions of positive homogeneity (5.65) and of sub-additivity (5.76).

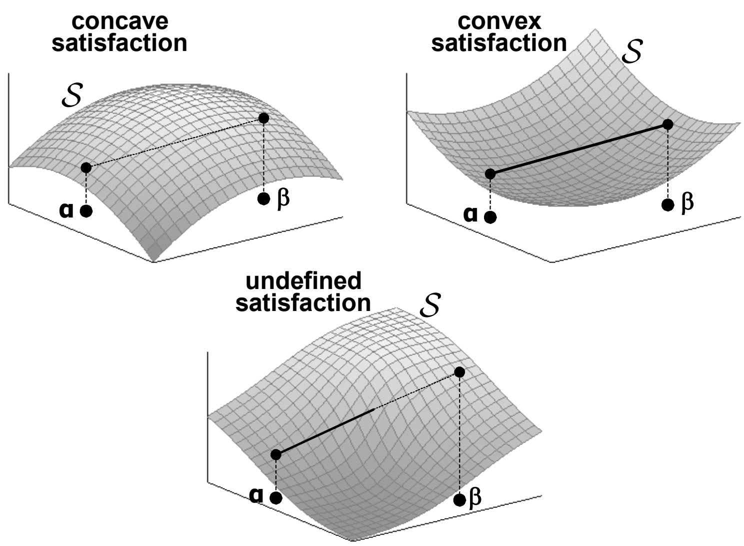

Fig. 5.6. Concavity/convexity of satisfaction index

We sketch in Figure 5.6 the geometrical interpretation of the above properties.

From a theoretical point of view, the most remarkable property of a concave index of satisfaction is the fact that such an index promotes diversification: the satisfaction derived by a diversified portfolio (the weighted average of two generic allocations) exceeds the average of the satisfaction derived by each portfolio individually. We stress that this property is independent of the market. In other words a concave index of satisfaction promotes diversification among two generic portfolios no matter their joint distribution: for example, the two portfolios might be highly positively or negatively correlated. Similarly, a convex index of satisfaction promotes concentration.

From a practical point of view, concavity is an important issue when we resort to numerical solutions to determine the best allocation, see Section 6.2.

- **Risk aversion/propensity/neutrality**

Loosely speaking, a measure of satisfaction is risk averse (risk seeking) if it rejects (welcomes) non-rewarded risk. More precisely, consider an allocation $\mathbf{b}$ that gives rise to a deterministic objective $\psi_{\mathbf{b}}$, i.e. an objective that is not a random variable. Consider now a *fair game*, i.e. an allocation $\mathbf{f}$ such that its objective $\Psi_{\mathbf{f}}$ has zero expected value.[2]

For example, a fair game is a bet on the outcome of tossing a coin: the investor wins a given amount of money if the outcome is "tail" and loses the same amount of money if the outcome is "head".

The joint allocation $\mathbf{b}+\mathbf{f}$ presents the investor with some risk, the fair game, which is not rewarded, since from (5.77) the expected value of the risky allocation is the same as the value of the risk-free allocation.

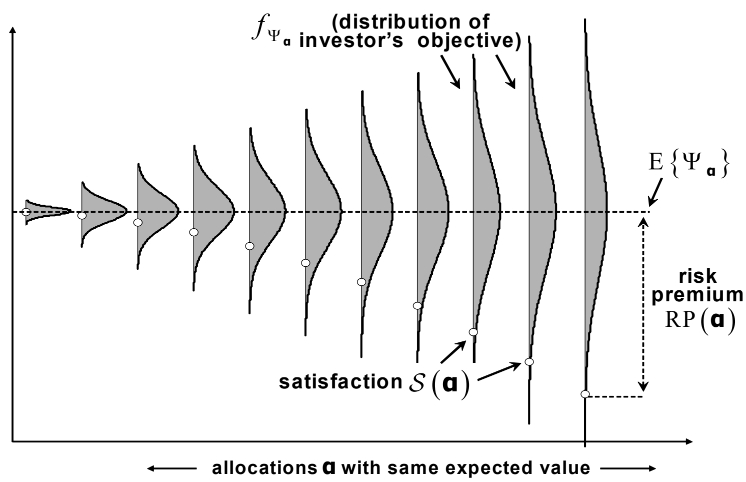

Fig. 5.7. Risk aversion and risk premium

An index of satisfaction is *risk averse* if the risk-free allocation $\mathbf{b}$ is preferred to the risky joint allocation $\mathbf{b}+\mathbf{f}$ for any level of the risk-free outcome $\psi_{\mathbf{b}}$ and any fair game $\mathbf{f}$:

$$\Psi_{\mathbf{b}} \equiv \psi_{\mathbf{b}},\ \mathrm{E}\{\Psi_{\mathbf{f}}\} \equiv 0 \ \Rightarrow\ \mathcal{S}(\mathbf{b}) \geq \mathcal{S}(\mathbf{b}+\mathbf{f}). \tag{5.83}$$

In words, the satisfaction of the risky joint allocation is less than the satisfaction of the deterministic allocation.

The *risk premium* is the dissatisfaction due to the uncertainty of a risky allocation:

$$\mathrm{RP} \equiv \mathcal{S}(\mathbf{b}) - \mathcal{S}(\mathbf{b}+\mathbf{f}). \tag{5.84}$$

[2] One can define a fair game in many different ways. We consider this definition because it is the most widely accepted in the financial literature.

If satisfaction is measured in terms of money, the risk premium is the compensation that the investor needs in order to make up for the uncertainty of his investment.

Any random variable Ψ for which the expected value is defined can be factored into the sum of a deterministic component $\mathrm{E}\{\Psi\}$ and a fair game $\Psi - \mathrm{E}\{\Psi\}$. Therefore we can define the risk premium associated with an allocation as the difference between the satisfaction arising from the expected objective and that arising from the risky allocation. In particular, if an index satisfies the constancy property (5.62) then the risk premium (5.84) associated with an allocation becomes:

$$\mathrm{RP}(\boldsymbol{\alpha}) \equiv \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} - \mathcal{S}(\boldsymbol{\alpha}). \tag{5.85}$$

In terms of the risk premium, an equivalent way to restate the definition of risk aversion (5.83) is the following: an index of satisfaction is *risk averse* if the risk premium is positive for any allocation:

$$\text{risk aversion: } \mathrm{RP}(\boldsymbol{\alpha}) \geq 0, \tag{5.86}$$

see Figure 5.7.

Similarly, an index of satisfaction is *risk seeking* if a risky allocation is preferred to a risk-free allocation with the same expected value. In other words, the risk premium associated with a risk seeking index of satisfaction is negative, as the investor is willing to pay a positive amount to play a risky game:

$$\text{risk propensity: } \mathrm{RP}(\boldsymbol{\alpha}) \leq 0. \tag{5.87}$$

Finally, an index of satisfaction is *risk neutral* if a risky allocation is perceived as equivalent to a risk-free allocation with the same expected value. In other words, the risk premium is zero:

$$\text{risk neutrality: } \mathrm{RP}(\boldsymbol{\alpha}) \equiv 0. \tag{5.88}$$

For example, the expected value of the objective (5.49) is trivially a risk neutral index of satisfaction:

$$\mathrm{RP}(\boldsymbol{\alpha}) \equiv \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} - \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} \equiv 0. \tag{5.89}$$

5.4 Certainty-equivalent (expected utility)

In Section 5.3 we supported the abstract discussion on indices of satisfaction with the example (5.49) of the expected value of the investor's objective. This example in practical applications is too simplistic. In this section we discuss the first of three broad and flexible classes of indices that allow us to model the investor's satisfaction in a variety of situations.

Consider an investor with a given objective Ψ such as absolute wealth, as in (5.3), or relative wealth, as in (5.4), or net profits, as in (5.8), or possibly other specifications. Consider a generic allocation $\boldsymbol{\alpha}$ that gives rise to the objective $\Psi_{\boldsymbol{\alpha}}$. A *utility function* $u(\psi)$ describes the extent to which the investor enjoys the generic outcome $\Psi_{\boldsymbol{\alpha}} = \psi$ of the objective, in case that realization takes place.

To build an index of satisfaction we can weight the utility from every possible outcome by the probability of that outcome. In other words, we consider the expected utility from the given allocation:

$$\boldsymbol{\alpha} \mapsto \mathrm{E}\left\{u\left(\Psi_{\boldsymbol{\alpha}}\right)\right\} \equiv \int_{\mathbb{R}} u(\psi)\, f_{\Psi_{\boldsymbol{\alpha}}}(\psi)\, d\psi, \tag{5.90}$$

where f_{Ψ} is the probability density function of the objective. This expression is in the form (5.48) and thus it qualifies as a potential index of satisfaction. Indeed, this is the *Von Neumann-Morgenstern* specification of expected utility as an index of satisfaction, see Varian (1992): the investor prefers an allocation that gives rise to a higher expected utility.

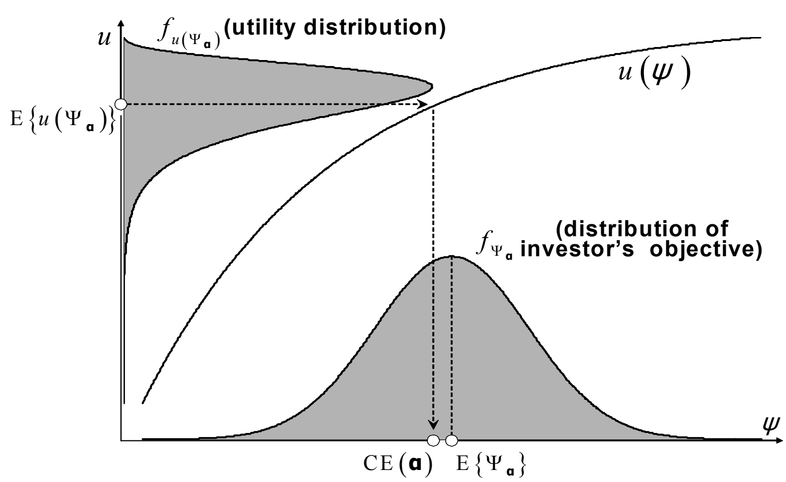

Fig. 5.8. Expected utility and certainty-equivalent

For example, consider the *exponential utility function*:

$$u(\psi) \equiv -e^{-\frac{1}{\zeta}\psi}, \tag{5.91}$$

where ζ is a constant. Since the objective has the dimensions of [money], in order to make the argument of the exponential function dimensionless, this constant must have the dimensions of [money]. The expected utility reads:

$$\mathrm{E}\left\{u\left(\Psi_{\boldsymbol{\alpha}}\right)\right\} = -\mathrm{E}\left\{e^{-\frac{1}{\zeta}\Psi_{\boldsymbol{\alpha}}}\right\} = -\phi_{\Psi_{\boldsymbol{\alpha}}}\left(\frac{i}{\zeta}\right), \tag{5.92}$$

where ϕ denotes the characteristic function (1.12) of the objective.

Nevertheless, utility cannot be measured in natural units (a meter of utility? a watt of utility?). Furthermore, for practitioners it is more intuitive to measure satisfaction in terms of money. In order to satisfy this requirement, we consider the *certainty-equivalent* of an allocation, which is the risk-free amount of money that would make the investor as satisfied as the risky allocation:

$$\boldsymbol{\alpha} \mapsto \mathrm{CE}\left(\boldsymbol{\alpha}\right) \equiv u^{-1}\left(\mathrm{E}\left\{u\left(\Psi_{\boldsymbol{\alpha}}\right)\right\}\right), \tag{5.93}$$

see Figure 5.8.

The certainty-equivalent is measured in the same units as the objective $\Psi_{\boldsymbol{\alpha}}$, which is money. Therefore, instead of (5.90), we choose the certainty-equivalent (5.93) as an index of satisfaction.

For example, consider the exponential utility function (5.91). From (5.92) the certainty-equivalent reads:

$$\boldsymbol{\alpha} \mapsto \mathrm{CE}\left(\boldsymbol{\alpha}\right) \equiv -\zeta \ln\left(\phi_{\Psi_{\boldsymbol{\alpha}}}\left(\frac{i}{\zeta}\right)\right). \tag{5.94}$$

Since ζ has the dimensions of [money] and the characteristic function is dimensionless, the certainty-equivalent has the dimensions of [money].

Notice that at this stage the symbol u^{-1} in (5.93) is a pseudo-inverse, i.e. just a notational convention to denote one, if any, solution to the following implicit equation:

$$u\left(\mathrm{CE}\left(\boldsymbol{\alpha}\right)\right) \equiv \mathrm{E}\left\{u\left(\Psi_{\boldsymbol{\alpha}}\right)\right\}. \tag{5.95}$$

Nevertheless, we will see below that the inverse is always defined and thus there always exists a unique certainty-equivalent.

5.4.1 Properties

In this section we revisit the properties of a generic index of satisfaction discussed in Section 5.3 to ascertain which are satisfied by the certainty-equivalent.

- **Money-equivalence**

By construction the certainty-equivalent (5.93) has the same dimensions as the objective and thus it is measured in terms of money. Therefore the certainty-equivalent is a money-equivalent index of satisfaction.

- **Estimability**

The certainty-equivalent is an estimable index of satisfaction. Indeed, from the formulation (5.90) we see that the expected utility of an allocation is a functional of the probability density function of the investor's objective. Since the certainty-equivalent is defined in terms of the expected utility as in (5.93), the certainty-equivalent is also a functional of the probability density function of the investor's objective. Therefore the certainty-equivalent is defined through a chain such as (5.52) and thus it is an estimable index of satisfaction:

$$\boldsymbol{\alpha} \mapsto \Psi_{\boldsymbol{\alpha}} \mapsto f_{\Psi_{\boldsymbol{\alpha}}} \mapsto \mathrm{CE}\left(\boldsymbol{\alpha}\right). \tag{5.96}$$

Depending on the situation, it might be more natural to represent the distribution of the objective equivalently in terms of the characteristic function or the cumulative distribution function.

For example, in the case of the exponential utility function (5.91) the certainty-equivalent (5.94) is defined through the following chain:

$$\boldsymbol{\alpha} \mapsto \Psi_{\boldsymbol{\alpha}} \mapsto \phi_{\Psi_{\boldsymbol{\alpha}}} \mapsto \mathrm{CE}\left(\boldsymbol{\alpha}\right) \equiv -\zeta \ln\left(\phi_{\Psi_{\boldsymbol{\alpha}}}\left(\frac{i}{\zeta}\right)\right), \tag{5.97}$$

which is in the form (5.52). Therefore the exponential certainty-equivalent is estimable.

- **Sensibility**

Due to the non-satiation principle, investors pursue the largest possible amount of their respective objectives. Therefore utility must be an increasing function of the objective. Assuming that the utility function is smooth, this corresponds to the condition that the first derivative of the utility be positive for all values in the range of the investor's objective:

$$\mathcal{D}u \geq 0, \tag{5.98}$$

where $\mathcal{D}$ is the derivative operator (B.25). This is the only, though essential, restriction that we impose on the utility function, and thus on the definition of the certainty-equivalent.

For example, consider the exponential utility function (5.91). In order for the utility to be an increasing function of the objective, we impose the constraint $\zeta > 0$.

The consequences of restricting the utility to the set of increasing functions are manifold.

In the first place, the Von Neumann-Morgenstern specification (5.90) and the certainty-equivalent specification (5.93) as indices of satisfaction become equivalent. In other words, if the utility function u is increasing, its inverse u^{-1} is well-defined and increasing. Therefore the certainty-equivalent is an increasing function of the expected utility and an allocation $\boldsymbol{\alpha}$ gives rise to a larger expected utility than an allocation $\boldsymbol{\beta}$ if and only if that allocation $\boldsymbol{\alpha}$ gives rise to a larger certainty-equivalent than $\boldsymbol{\beta}$:

$$\mathrm{E}\{u(\Psi_{\boldsymbol{\alpha}})\} \geq \mathrm{E}\{u(\Psi_{\boldsymbol{\beta}})\} \Leftrightarrow \mathrm{CE}(\boldsymbol{\alpha}) \geq \mathrm{CE}(\boldsymbol{\beta}). \tag{5.99}$$

Secondly, if the utility function is increasing, the certainty-equivalent is sensible, i.e. consistent with strong dominance. Indeed, from (5.99) and the fact that u is increasing we derive:

$$\Psi_{\boldsymbol{\alpha}} \geq \Psi_{\boldsymbol{\beta}} \text{ in all scenarios } \Rightarrow \mathrm{CE}(\boldsymbol{\alpha}) \geq \mathrm{CE}(\boldsymbol{\beta}), \tag{5.100}$$

which is the definition (5.55) of sensibility applied to the certainty-equivalent.

Finally, we can relate the utility function to the investor's *subjective probability*, see Castagnoli and LiCalzi (1996) and Bordley and LiCalzi (2000).

Suppose that the investor has an a-priori hunch as to how his investment will perform, no matter the actual investment decision. We can describe this hunch in terms of a subjective distribution of the objective Ψ, whose pdf (cdf) we denote as f_{Ψ}^{S} (F_{Ψ}^{S}).

Now consider a specific allocation $\boldsymbol{\alpha}$. We can compare this allocation with the investor's hunch by means of (5.42). In other words, we consider the variable:

$$W(V) \equiv F_{\Psi}^{\mathrm{S}}\left(Q_{\Psi_{\boldsymbol{\alpha}}}(V)\right), \tag{5.101}$$

where $V \sim \mathrm{U}([0,1])$.

If the joint outcomes of V and W plot above the diagonal as in Figure 5.3, the investor is a pessimist. Indeed in this case the objective $\boldsymbol{\alpha}$ weakly dominates the investor's hunch, which means that the allocation $\boldsymbol{\alpha}$ is better than the investor thinks. On the other hand, if the joint outcomes of V and W plot below the diagonal, the investor is an optimist, as the allocation $\boldsymbol{\alpha}$ is worse than the investor thinks.

No matter the degree of optimism of the investor, the higher the graph, the better the investment. Therefore the investor's satisfaction is an increasing function of the "aboveness" of the graph (5.101). To quantify the degree of "aboveness" of the graph, the most natural approach is to compute its expected value:

$$\begin{aligned} \mathrm{E}\{W(V)\} &\equiv \int_0^1 F_{\Psi}^{\mathrm{S}}\left(Q_{\Psi_{\boldsymbol{\alpha}}}(v)\right) dv \\ &= \int_{\mathbb{R}} F_{\Psi}^{\mathrm{S}}(\psi)\, f_{\Psi_{\boldsymbol{\alpha}}}(\psi)\, d\psi. \end{aligned} \tag{5.102}$$

Comparing this expression with (5.90) we see that the utility function u represents the cumulative distribution function F_{Ψ}^{S} of the investor's subjective a-priori hunch on the result of his investments.

For this interpretation to make sense, the utility function must be increasing and satisfy the normalization properties (1.10) of any cumulative distribution function:

$$u(\psi_{\inf}) \equiv 0, \quad u(\psi_{\sup}) \equiv 1, \tag{5.103}$$

where $[\psi_{\inf}, \psi_{\sup}]$ is the (possibly unbounded) domain of the utility function, i.e. the range of the investor's objective.

Although (5.103) seems a very restrictive assumption on the utility, we notice that the certainty-equivalent is unaffected by positive affine transformations of the utility function. In other words, if in (5.95) we shift and stretch the investor's utility function as follows:

$$u(\psi) \mapsto a + bu(\psi), \tag{5.104}$$

where b is a positive number, the certainty-equivalent is unaffected. Therefore we can always normalize any utility function in such a way that (5.103) holds.[3]

For example, assume that the investor's objective is positive: $\psi \in [0, +\infty)$ and consider the exponential utility function (5.91). This function is equivalent to:

$$u(\psi) \equiv 1 - e^{-\frac{1}{\zeta}\psi}, \tag{5.105}$$

which satisfies (5.103) if $\zeta > 0$. The first derivative of this expression yields the probability density function of the investor's subjective view on his investments:

$$f_{\Psi}^{\mathrm{S}}(\psi) = \frac{1}{\zeta} e^{-\frac{1}{\zeta}\psi}. \tag{5.106}$$

This is a decreasing function: in other words, an investor whose utility is exponential is a pessimist who believes that the worst scenarios are the most likely to occur.

We remark that the interpretation of the utility function u as a subjective cumulative distribution function F_{Ψ}^{S} reduces the certainty-equivalent to a quantile-based index of satisfaction like the value at risk, see Section 5.5. Indeed, denoting as Q_{Ψ}^{S} the quantile of the subjective distribution of the objective, from the definition of the certainty-equivalent (5.93) we obtain:

$$\mathrm{CE}(\boldsymbol{\alpha}) = Q_{\Psi}^{\mathrm{S}}(c_{\boldsymbol{\alpha}}), \tag{5.107}$$

where the confidence level reads:

$$c_{\boldsymbol{\alpha}} \equiv \mathrm{E}\left\{F_{\Psi}^{\mathrm{S}}(\Psi_{\boldsymbol{\alpha}})\right\}. \tag{5.108}$$

In words, the certainty-equivalent is the quantile of the investor's subjective distribution, where the confidence level is the expected subjective grade.

[3] If the domain $[\pi_{\inf}, \pi_{\sup}]$ is unbounded, the utility function should be bounded for the normalization to make sense. If this is not the case, we can overcome this problem by restricting $[\pi_{\inf}, \pi_{\sup}]$ to a bounded, yet arbitrarily large, domain.

- **Consistence with stochastic dominance**

To guarantee that the certainty-equivalent (5.93) is a sensible index of satisfaction we imposed the condition that the utility function be increasing. This condition also implies that the certainty-equivalent is consistent with weak dominance. In other words, if u is increasing then the following implication holds true:

$$Q_{\Psi_{\alpha}}(p) \geq Q_{\Psi_{\beta}}(p) \text{ for all } p \in (0,1) \Rightarrow \mathrm{CE}(\boldsymbol{\alpha}) \geq \mathrm{CE}(\boldsymbol{\beta}), \tag{5.109}$$

which is (5.57) in this context. This follows from (5.59).

Consistence of the certainty-equivalent with second-order dominance is guaranteed if the utility function is increasing and concave. Assuming that the utility is a smooth function, these conditions can be stated in terms of the derivative operator (B.25) as follows:

$$\mathcal{D}u \geq 0, \quad \mathcal{D}^2 u \leq 0. \tag{5.110}$$

In general, as far as consistence with higher order dominance is concerned, the certainty-equivalent is consistent with q-th order stochastic dominance if the following condition holds on the whole range of the investor's objective:

$$(-1)^k \mathcal{D}^k u \leq 0, \quad k = 1, 2, \ldots, q, \tag{5.111}$$

see Ingersoll (1987). A comparison of this condition with (5.98) and (5.110) shows that this result includes consistency with weak (first-order) and second-order stochastic dominance respectively.

- **Constancy**

If the investor's objective is deterministic, the certainty-equivalent (5.93) coincides with the objective. In other words, the certainty-equivalent satisfies the constancy requirement:

$$\Psi_{\mathbf{b}} \equiv \psi_{\mathbf{b}} \Rightarrow \mathrm{CE}(\mathbf{b}) = \psi_{\mathbf{b}}, \tag{5.112}$$

which is (5.62) in this context.

- **Positive homogeneity**

In order for the certainty-equivalent (5.93) to be a positive homogeneous index of satisfaction it has to satisfy (5.65), which in this context reads:

$$\mathrm{CE}(\lambda \boldsymbol{\alpha}) = \lambda \, \mathrm{CE}(\boldsymbol{\alpha}). \tag{5.113}$$

In Appendix www.5.3 we show that the class of utility functions that gives rise to a positive homogeneous certainty-equivalent is the power class:

$$u(\psi) \equiv \psi^{1-\frac{1}{\gamma}}, \tag{5.114}$$

where $\gamma \geq 1$. Indeed, Figure 5.4 was generated using the certainty-equivalent relative to a power utility function.

We discuss this type of utility in a more general context below, see (5.135). In particular, if the utility function is of the power class, the Euler decomposition (5.67) of the certainty-equivalent holds, see (5.152) below.

- **Translation invariance**

In order for the certainty-equivalent (5.93) to be a translation invariant index of satisfaction it has to satisfy (5.72), which in this context reads:

$$\Psi_{\mathbf{b}} \equiv 1 \Rightarrow \mathrm{CE}\left(\boldsymbol{\alpha}+\lambda \mathbf{b}\right)=\mathrm{CE}\left(\boldsymbol{\alpha}\right)+\lambda. \tag{5.115}$$

In Appendix www.5.3 we show that the class of utility functions that give rise to a translation invariant certainty-equivalent is the exponential class (5.91). Indeed, Figure 5.5 was generated using the certainty-equivalent relative to an exponential utility function. We discuss this type of utility function in a more general context below, see (5.133).

- **Super-/sub- additivity**

The certainty-equivalent (5.93) is not a super-additive index of satisfaction. For this to be the case, the certainty-equivalent should satisfy the following relation for any two allocations $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$:

$$\mathrm{CE}\left(\boldsymbol{\alpha}+\boldsymbol{\beta}\right) \geq \mathrm{CE}\left(\boldsymbol{\alpha}\right)+\mathrm{CE}\left(\boldsymbol{\beta}\right), \tag{5.116}$$

which is (5.75) in this context. The only utility function such that (5.116) is true no matter the market distribution is the linear utility, in which case the certainty-equivalent becomes the expected value of the objective:

$$u\left(\psi\right) \equiv \psi \Leftrightarrow \mathrm{CE}\left(\boldsymbol{\alpha}\right)=\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\}. \tag{5.117}$$

In this situation from (5.77) the certainty-equivalent is additive, and thus (5.116) holds as an equality.

Similarly, the certainty-equivalent is not a sub-additive index of satisfaction unless the utility is linear.

- **Co-monotonic additivity**

The certainty-equivalent (5.93) is not a co-monotonic additive index of satisfaction. In other words, if an allocation $\boldsymbol{\delta}$ gives rise to an objective that is an increasing function of the objective of another allocation $\boldsymbol{\alpha}$ as in (2.35), the certainty-equivalent of the total portfolio is not necessarily the sum of the certainty-equivalents of the separate allocations:

$$\left(\boldsymbol{\alpha},\boldsymbol{\delta}\right) \text{ co-monotonic } \nRightarrow \mathrm{CE}\left(\boldsymbol{\alpha}+\boldsymbol{\delta}\right)=\mathrm{CE}\left(\boldsymbol{\alpha}\right)+\mathrm{CE}\left(\boldsymbol{\delta}\right). \tag{5.118}$$

This can be proved easily with a counterexample. The only utility function such that (5.118) is true no matter the market distribution is the linear utility $u\left(\psi\right) \equiv \psi$, in which case the co-monotonicity is a consequence of the additivity property (5.77).

- **Concavity/convexity**

One might think that the concavity/convexity properties of the certainty-equivalent are a straightforward consequence of the concavity/convexity properties of the utility function. This is not the case.

Intuitively, in the definition of the certainty-equivalent (5.93) the expected utility is a concave function of $\boldsymbol{\alpha}$ if and only if the utility function is concave. On the other hand, if u is concave, the inverse u^{-1} in the definition of the certainty-equivalent is convex, and the two effects tend to cancel each other.

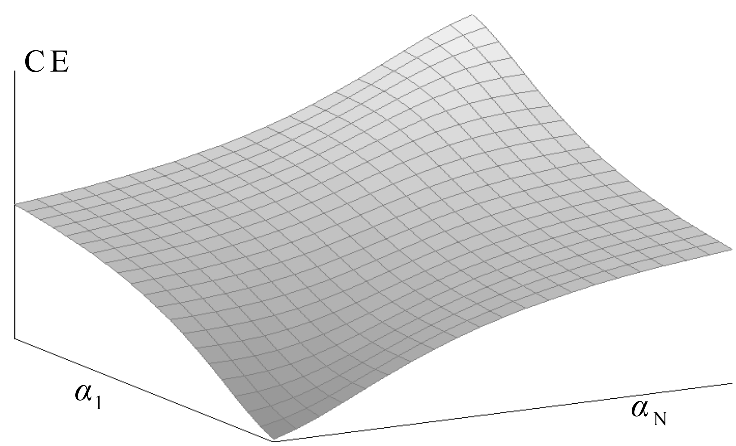

Fig. 5.9. Certainty equivalent as function of allocation

Therefore, in general the certainty-equivalent is neither concave nor convex, see Figure 5.9. We present this argument more formally in the context of the second-order sensitivity analysis in Section 5.4.4.

- **Risk aversion/propensity**

The certainty-equivalent (5.93) satisfies the constancy property, see (5.112). Therefore the risk premium associated with an allocation is given by (5.85), which in this context reads:

$$\mathrm{RP}(\boldsymbol{\alpha}) = \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} - \mathrm{CE}(\boldsymbol{\alpha}). \tag{5.119}$$

In Appendix www.5.3 we prove that for any allocation $\boldsymbol{\alpha}$ the following result holds:

$$u \text{ concave } \Leftrightarrow \mathrm{RP}(\boldsymbol{\alpha}) \geq 0. \tag{5.120}$$

Therefore from (5.86) the certainty-equivalent is a risk averse index of satisfaction if and only if the utility function is concave. This is the situation for instance in Figure 5.8, where (5.120) is satisfied.

Similarly, the certainty-equivalent is a risk prone index of satisfaction if and only if the utility function is convex. Finally, the certainty-equivalent is a risk neutral index of satisfaction if and only if the utility function is linear.

Risk aversion is a global feature: the risk premium of a risk averse investor is positive no matter the allocation. Similarly, risk propensity is a global feature. Nevertheless, depending on their objectives and conditions, investors might display different attitudes toward risk in different situations, and thus the risk premium they require can change sign.

For instance, prospect theory asserts that the investor's objective are the net profits as in (5.8). In this context, investors tend to be cautious in their pursuit of new gains, but are unwilling to cut their losses in the hope of a recovery, see Kahneman and Tversky (1979). The ensuing utility function is S-shaped, i.e. it is concave (= risk averse) for profits and convex (= risk prone) for losses, see the fourth plot in Figure 5.10.

To better describe the investor's attitude toward risk we need a more local measure. The *Arrow-Pratt absolute risk aversion* is defined as follows:

$$\mathrm{A}(\psi) \equiv -\frac{\mathcal{D}^2 u(\psi)}{\mathcal{D} u(\psi)}, \tag{5.121}$$

where $\mathcal{D}$ is the derivative operator (B.25). In Appendix www.5.3 we show that if an allocation $\boldsymbol{\alpha}$ gives rise to an objective which is not too volatile, i.e. an objective whose distribution is highly concentrated around its expected value, the following factorization yields a good approximation of the risk premium:

$$\mathrm{RP}(\boldsymbol{\alpha}) \approx \frac{1}{2}\mathrm{A}(\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\})\,\mathrm{Var}\{\Psi_{\boldsymbol{\alpha}}\}. \tag{5.122}$$

In other words, the money necessary to compensate for the riskiness of an allocation is the product of a quantity that depends on the investor's preferences, namely the Arrow-Pratt risk aversion, and a quantity that does not depend on the investor's preferences, namely the variance of the allocation, which in turn summarizes its riskiness.

The Arrow-Pratt risk aversion (5.121) is a function: as such, it is a local measure of risk aversion that depends on the expected value of the objective. Since the first derivative of the utility function is always positive, the Arrow-Pratt risk aversion is positive if and only if the second derivative of the utility function is negative, which means that the utility function is concave. In other words, from (5.120) the Arrow-Pratt risk aversion is positive if and only if the investor is locally risk averse. Similarly, the Arrow-Pratt risk aversion is negative if and only if the investor is locally risk prone.

For example, assume a prospect-theory framework where the objective are net profits as in (5.8). We use the error function (B.75) to model the investor's utility:

$$u(\psi) \equiv \operatorname{erf}\left(\frac{\psi}{\sqrt{2\eta}}\right). \tag{5.123}$$

In this expression η is a constant with the dimensions of $[\text{money}]^2$ that makes the argument of the error function dimensionless. The error function is S-shaped, with a flex point in zero, see the fourth plot in Figure 5.10.

The domain of this utility function is the whole real axis: this function describes the attitude towards risk in the case of both profits, i.e. positive values of the objective, and losses, i.e. negative values of the objective. The Arrow-Pratt risk aversion (5.121) corresponding to the error function utility (5.123) reads:

$$\mathrm{A}(\psi) = \frac{\psi}{\eta}. \tag{5.124}$$

Consider the case where η is positive. When facing net gains, i.e. when ψ is positive, the Arrow-Pratt risk aversion is positive: the investor is risk averse and seeks a compensation for the non-rewarded risk. On the other hand, when facing net losses, i.e. when ψ is negative, the Arrow-Pratt risk aversion is negative: the investor is risk prone and is willing to pay a premium to hold on to a risky allocation.

Notice that the Arrow-Pratt risk aversion is not a dimensionless number: instead, it as has the dimensions of $[\text{money}]^{-1}$. This follows from the definition (5.121) and is a necessary condition for the dimensional consistency of (5.122).

For example, in (5.124) the objective has the dimensions of [money] and the constant η has the dimensions of $[\text{money}]^2$. Therefore their ratio has the dimensions of $[\text{money}]^{-1}$.

Unlike the second derivative of the utility function, not only the sign, but also the absolute value of the Arrow-Pratt risk aversion is meaningful. Indeed, from (5.104) any utility function is defined only modulo positive affine transformations: such transformations affect the value of the first and second derivative of the utility function, but leave the Arrow-Pratt risk aversion (5.121) unaltered. Alternatively, the fact that the Arrow-Pratt risk aversion has the dimensions of $[\text{money}]^{-1}$ is proof of its importance.

5.4.2 Building utility functions

According to the certainty-equivalent approach, the attitude toward risk of an investor is described by his utility function. Therefore we should use a tailor-made utility function for each specific case. Since this is impossible, we specify the functional form of the utility function by means of parsimonious yet flexible parametrizations. We present below two methods to build parametric utility functions.

Basis

One way to build utility functions starts by specifying a *basis*. In other words, we specify a one-parameter set of (generalized) functions b that can generate a whole class of utility functions as weighted averages:

$$u(\psi) \equiv \int_{\mathbb{R}} g(\theta) b(\theta, \psi) d\theta, \tag{5.125}$$

where the weight function g is positive and sums to one:

$$g \geq 0, \quad \int_{\mathbb{R}} g(\theta) d\theta = 1. \tag{5.126}$$

For example, consider a basis defined in terms of the Heaviside function ($B.74$) as follows:

$$b(\theta, \psi) \equiv H^{(\theta)}(\psi). \tag{5.127}$$

This basis is very broad, as it can in principle generate all sensible utility functions. Indeed, any increasing function u can be expressed in terms of this basis as in (5.125), if the weight function is defined as the derivative of the utility function:

$$g \equiv \mathcal{D}u. \tag{5.128}$$

The proof follows immediately from the definition of the Heaviside step function. Also notice that the condition (5.103) guarantees that the normalization (5.126) is satisfied.

The representation of utility functions in terms of a basis is useful in two ways. In the first place, it allows us to build classes of utility functions that share the same properties.

For example, consider the min-function basis:

$$b(\theta, \psi) \equiv \min(\psi, \theta). \tag{5.129}$$

This basis can generate all concave utility functions. Therefore the min-function basis can be used to describe risk averse investors, see Gollier (2001).

Secondly, the representation of utility functions in terms of a basis provides a probabilistic interpretation of the expected utility and therefore of the certainty-equivalent:

$$\begin{aligned} \mathrm{E}\{u(\Psi)\} &= \int_{\mathbb{R}^2} b(\theta, \psi) g(\theta) f_{\Psi}(\psi) d\theta d\psi \\ &= \mathrm{E}\{b(\Theta, \Psi)\}. \end{aligned} \tag{5.130}$$

In other words, the expected utility is the expected value of a function of two random variables. The first random variable Θ models the investor's preferences and attitude toward risk. The second random variable Ψ, i.e. the objective, models the market. These two random variables are independent and thus fully determined by their marginal distributions. The distribution of the preferences is described by the weight function g, which due to (5.126) can be interpreted as a probability density function; the distribution of the market is described by the probability density function of the investor's objective f_{Ψ}.

Arrow-Pratt risk aversion

A different approach to building utility functions focuses on the Arrow-Pratt risk aversion.

First of all notice that the Arrow-Pratt risk aversion is an equivalent, yet more efficient, representation of the utility function. Indeed, from (5.121) the specification of the utility function yields the Arrow-Pratt risk aversion. In turn, integrating the Arrow-Pratt risk aversion, we can recover the utility function, modulo a positive affine transformation:

$$(\mathcal{I} \circ \exp \circ \mathcal{I})[-\mathrm{A}] = a + bu, \tag{5.131}$$

where $\mathcal{I}$ is the integration operator $(B.27)$. From (5.104), positive affine transformations are irrelevant to the determination of the certainty-equivalent: therefore, the Arrow-Pratt risk aversion contains all and only the information about the utility function that matters in determining the investor's satisfaction.

Therefore, an efficient way to define parametric forms of the investor's preferences is to specify flexible, although parsimonious, functional forms for the Arrow-Pratt risk aversion, rather than for the utility function. A possible such specification appears in LiCalzi and Sorato (2003):

$$\mathrm{A}(\psi) \equiv \frac{\psi}{\gamma \psi^2 + \zeta \psi + \eta}. \tag{5.132}$$

This specification depends on only three constant parameters[4] and yet it includes as special cases most of the parametrizations studied in the financial literature. Since Pearson (1895) discussed a similar parametrization in a different context, (5.132) is called the *Pearson specification* of the utility function.

In the special case where $\eta \equiv 0$ in (5.132) we obtain the *Hyperbolic Absolute Risk Aversion (HARA)* class of utility functions. The HARA class includes in turn a few notable parametrizations as special cases. The results below can be obtained by applying (5.131) or by checking the derivatives of the utility function in the definition (5.121) of the Arrow-Pratt risk aversion coefficient.

If $\zeta > 0$ and $\gamma \equiv 0$ the HARA class yields the *exponential utility*:

$$u(\psi) = -e^{-\frac{1}{\zeta}\psi}, \tag{5.133}$$

where $\zeta > 0$, see Figure 5.10.

[4] By adding one more parameter ξ as follows $\mathrm{A}(\psi, \xi) \equiv \mathrm{A}(\psi - \xi)$, all the ensuing utility functions are shifted along the horizontal axis by ξ.

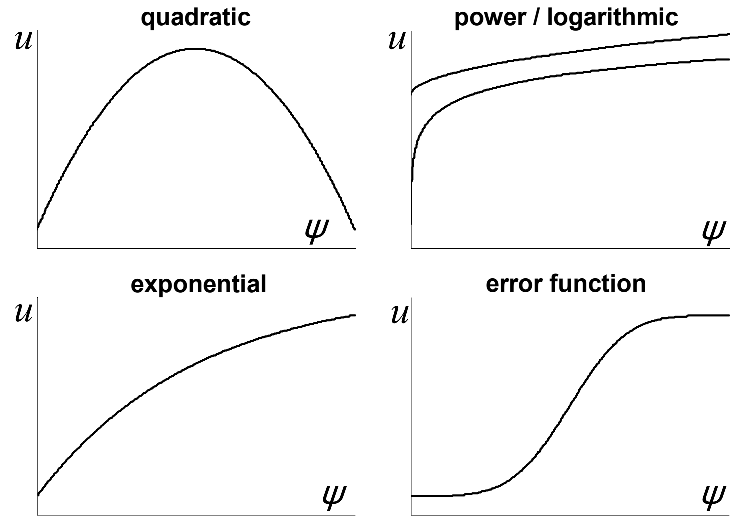

Fig. 5.10. Parametric utility functions

If $\zeta > 0$ and $\gamma \equiv -1$ the HARA class yields the *quadratic utility*:

$$u(\psi) = \psi - \frac{1}{2\zeta}\psi^2. \tag{5.134}$$

We remark that the certainty-equivalent stemming from the quadratic utility function is not sensible for $\psi > \zeta$ because in that region a larger value of the objective ψ decreases the investor's satisfaction: therefore this utility function can be used in principle only when the objective is bounded from above, see Figure 5.10.

If $\zeta \equiv 0$ and $\gamma \geq 1$ the HARA class yields the *power utility*:

$$u(\psi) \equiv \psi^{1-\frac{1}{\gamma}}, \tag{5.135}$$

see Figure 5.10. In the limit $\gamma \to 1$ the power utility (5.135) yields, modulo a positive affine transformation, the *logarithmic utility* function:

$$u(\psi) \equiv \ln \psi. \tag{5.136}$$

In the limit $\gamma \to \infty$ the power utility (5.135) becomes the *linear utility*:

$$u(\psi) \equiv \psi. \tag{5.137}$$

The utility functions in the HARA class are very flexible, but always concave. Therefore the HARA specification cannot properly model the framework of *prospect theory*, where the objective are the net profits and the utility function must be S-shaped. Nevertheless, the more general Pearson specification (5.132) allows such flexibility. To see this in a specific case, we set $\gamma \equiv 0$ and $\zeta \equiv 0$, obtaining the *error function utility*:

$$u(\psi) \equiv \operatorname{erf}\left(\frac{\psi}{\sqrt{2\eta}}\right), \tag{5.138}$$

see Figure 5.10.

5.4.3 Explicit dependence on allocation

We recall from (5.10) that the investor's objective, namely absolute wealth, relative wealth, net profits, or other specifications, is a simple linear function of the allocation and the market vector:

$$\Psi_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'\mathbf{M}. \tag{5.139}$$

Therefore the certainty-equivalent (5.93) depends on the allocation as follows:

$$\boldsymbol{\alpha} \mapsto \mathrm{CE}(\boldsymbol{\alpha}) \equiv u^{-1}\left(\mathrm{E}\left\{u\left(\boldsymbol{\alpha}'\mathbf{M}\right)\right\}\right). \tag{5.140}$$

In this section we tackle the problem of computing explicitly the certainty-equivalent of an allocation for a given distribution of the market vector $\mathbf{M}$ and a given choice of the utility function u.

For very special combinations of the distribution of the market and of the choice of the utility functions the dependence of the certainty-equivalent on the allocation can be computed analytically.

For example, we can represent the distribution of the market in terms of its characteristic function $\phi_{\mathbf{M}}$. Then the distribution of the investor's objective (5.139) is represented in terms of its characteristic function as follows:

$$\phi_{\Psi_{\alpha}}(\omega) = \phi_{\mathbf{M}}(\omega\boldsymbol{\alpha}), \tag{5.141}$$

see Appendix www.2.4.

Suppose that the investor's satisfaction is determined by an exponential utility function (5.91). Then from (5.94) the explicit dependence of the certainty-equivalent on allocation reads:

$$\mathrm{CE}(\boldsymbol{\alpha}) = -\zeta \ln\left(\phi_{\mathbf{M}}\left(\frac{i}{\zeta}\boldsymbol{\alpha}\right)\right). \tag{5.142}$$

Consider a market which at the investment horizon is normally distributed and assume that the investor's objective is final wealth. From (5.11) and (5.12) the market vector is normally distributed:

$$\mathbf{M} \equiv \mathbf{P}_{T+\tau} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{5.143}$$

From (5.142) and the characteristic function of the normal distribution (2.157) we obtain:

$$\boldsymbol{\alpha} \mapsto \mathrm{CE}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}'\boldsymbol{\mu} - \frac{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}{2\zeta}. \tag{5.144}$$

Notice that the expected value of the market prices $\boldsymbol{\mu}$ has the dimensions of [money] and the covariance matrix of the market prices $\boldsymbol{\Sigma}$ has the dimensions of [money]^2. The allocation vector $\boldsymbol{\alpha}$ is a dimensionless number. Since ζ has the dimensions of [money], so does the certainty-equivalent.

Nevertheless, for a generic utility function u and a generic market $\mathbf{M}$ the certainty-equivalent is a complex expression of the objective.

For example, consider a market for which the gamma approximation (5.24) holds, and an investor whose utility function is exponential. In this case from (5.94) and the expression of the characteristic function of the objective (5.30) we obtain:

$$\begin{aligned}\mathrm{CE}(\boldsymbol{\alpha}) = {} & \frac{\zeta}{2}\ln\left|\mathbf{I}_K + \frac{1}{\zeta}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}}\boldsymbol{\Sigma}\right| + \left(\theta_{\boldsymbol{\alpha}} + \boldsymbol{\Delta}'_{\boldsymbol{\alpha}}\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{\mu}'\boldsymbol{\Gamma}_{\boldsymbol{\alpha}}\boldsymbol{\mu}\right) \\ & + \frac{\zeta}{2}\left[\boldsymbol{\Delta}_{\boldsymbol{\alpha}} + \boldsymbol{\Gamma}_{\boldsymbol{\alpha}}\boldsymbol{\mu}\right]'\boldsymbol{\Sigma}\left(\mathbf{I}_K + \frac{1}{\zeta}\boldsymbol{\Gamma}_{\boldsymbol{\alpha}}\boldsymbol{\Sigma}\right)^{-1}\left[\boldsymbol{\Delta}_{\boldsymbol{\alpha}} + \boldsymbol{\Gamma}_{\boldsymbol{\alpha}}\boldsymbol{\mu}\right],\end{aligned} \tag{5.145}$$

where the explicit dependence on the allocation $\boldsymbol{\alpha}$ is given by (5.26)-(5.28).

When the explicit dependence of the certainty-equivalent on the allocation vector $\boldsymbol{\alpha}$ cannot be computed analytically, we can gain insight on this dependence by means of a first-order approximate expression, the *Arrow-Pratt approximation*:

$$\mathrm{CE}(\boldsymbol{\alpha}) \approx \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} - \frac{\mathrm{A}(\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\})}{2}\mathrm{Var}\{\Psi_{\boldsymbol{\alpha}}\}. \tag{5.146}$$

In this expression A is the Arrow-Pratt coefficient of risk aversion (5.121), see Appendix www.5.3 for a proof. This approximation shows that high expected values of the objective are always appreciated, whereas the investor's attitude towards the variance of its objective can vary, depending on the investor's local risk aversion.

If the moments of the market vector $\mathbf{M}$ are known, (5.146) yields the (approximate) explicit dependence of the certainty-equivalent on the allocation vector $\boldsymbol{\alpha}$. This follows from (5.139) and the affine equivariance properties (2.56) and (2.71) of expected value and covariance respectively.

If the moments of the market vector $\mathbf{M}$ are not known, we can replace the objective $\Psi_{\boldsymbol{\alpha}}$ with its gamma approximation $\Xi_{\boldsymbol{\alpha}}$ as in (5.25). This corresponds to replacing the moments of $\Psi_{\boldsymbol{\alpha}}$ that appear in (5.146) with the moments of $\Xi_{\boldsymbol{\alpha}}$, which can be computed from the derivatives of the characteristic function (5.30) as illustrated in Appendix www.5.1.

In the very special case of exponential utility and normal markets the approximation (5.146) becomes exact, as we see from (5.144).

5.4.4 Sensitivity analysis

Suppose that the investor has already chosen an allocation $\boldsymbol{\alpha}$ which yields a level of satisfaction $\mathrm{CE}(\boldsymbol{\alpha})$ and that he is interested in rebalancing his portfolio marginally by means of a small change $\delta\boldsymbol{\alpha}$ in the current allocation. In this case a local analysis in terms of a Taylor expansion is useful:

$$\begin{aligned} \mathrm{CE}(\boldsymbol{\alpha}+\delta\boldsymbol{\alpha}) \approx \mathrm{CE}(\boldsymbol{\alpha}) &+ \delta\boldsymbol{\alpha}' \frac{\partial \, \mathrm{CE}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} \\ &+ \frac{1}{2}\delta\boldsymbol{\alpha}' \frac{\partial^2 \, \mathrm{CE}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\alpha}'} \delta\boldsymbol{\alpha}. \end{aligned} \tag{5.147}$$

We prove in Appendix www.5.3 that the first-order derivatives read:

$$\frac{\partial \, \mathrm{CE}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} = \frac{\mathrm{E}\{\mathcal{D}u(\boldsymbol{\alpha}'\mathbf{M})\,\mathbf{M}\}}{\mathcal{D}u(\mathrm{CE}(\boldsymbol{\alpha}))}, \tag{5.148}$$

where $\mathcal{D}$ is the derivative operator $(B.25)$ and $\mathbf{M}$ is the random vector (5.139) that represents the market. The investor will focus on the entries of the vector (5.148) that display a large absolute value.

For example, consider a prospect theory setting where the investor's objective are the net profits. We model the investor's utility by means of the error function:

$$u(\psi) \equiv \operatorname{erf}\left(\frac{\psi}{\sqrt{2}\eta}\right). \tag{5.149}$$

Assume a normally distributed market:

$$\mathbf{P}_{T+\tau} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{5.150}$$

We prove in Appendix www.5.3 that the sensitivity of the certainty-equivalent to the allocation reads:

$$\frac{\partial \, \mathrm{CE}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} = \gamma(\boldsymbol{\alpha}) \left[\frac{1}{\eta}\boldsymbol{\alpha}\boldsymbol{\alpha}' + \boldsymbol{\Sigma}^{-1}\right]^{-1} \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{p}_T), \tag{5.151}$$

where $\gamma(\boldsymbol{\alpha})$ is a scalar and thus equally affects all the entries of the vector of the first derivatives.

We remark that when the utility function belongs to the power class (5.114) the certainty-equivalent is positive homogeneous and thus it can be expressed in terms of the contribution to satisfaction from each security by means of the Euler decomposition (5.67). Substituting the expression of the derivative of power utility function in (5.148) we obtain:

$$u(\psi) \equiv \psi^{1-\frac{1}{\gamma}} \Rightarrow \tag{5.152}$$
$$\mathrm{CE}(\boldsymbol{\alpha}) = \sum_{n=1}^{N} \alpha_n \left[\mathrm{E}\left\{ M_n (\boldsymbol{\alpha}'\mathbf{M})^{-\frac{1}{\gamma}} \right\} (\mathrm{CE}(\boldsymbol{\alpha}))^{\frac{1}{\gamma}} \right].$$

The contribution to satisfaction from each security in turn factors into the product of the amount of that security times the marginal contribution to satisfaction from that security. The marginal contribution to satisfaction, which is the term in square brackets in (5.152), is insensitive to a rescaling of the portfolio, although it depends on the allocation.

The study of the second-order cross-derivatives provides insight on the local convexity/concavity of the certainty-equivalent. In Appendix www.5.3 we prove the following result:

$$\frac{\partial^2 \mathrm{CE}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\alpha}} = \frac{\mathrm{E}\left\{ \mathcal{D}^2 u(\Psi_{\boldsymbol{\alpha}}) \mathbf{M}\mathbf{M}' \right\} - \mathcal{D}^2 u(\mathrm{CE}(\boldsymbol{\alpha})) \mathbf{w}\mathbf{w}'}{\mathcal{D} u(\mathrm{CE}(\boldsymbol{\alpha}))}, \tag{5.153}$$

where $\mathcal{D}$ is the derivative operator $(B.25)$ and the deterministic vector $\mathbf{w}$ is defined as follows:

$$\mathbf{w} \equiv \mathrm{E}\left\{ \frac{\mathcal{D} u(\Psi_{\boldsymbol{\alpha}})}{\mathcal{D} u(\mathrm{CE}(\boldsymbol{\alpha}))} \mathbf{M} \right\}. \tag{5.154}$$

In (5.153) the matrices $\mathbf{M}\mathbf{M}'$ and $\mathbf{w}\mathbf{w}'$are always positive, and so is $\mathcal{D}u$ by assumption. Nevertheless, even when the sign of $\mathcal{D}^2 u$ is consistently either negative or positive, we cannot be sure of the sign of the cross-derivatives (5.153). Therefore, the certainty-equivalent is neither a concave nor a convex index of satisfaction. We can see this in Figure 5.9, that refers to a prospect theory setting where utility is modeled by the error function (5.138).

5.5 Quantile (value at risk)

In Section 5.4 we discussed the certainty-equivalent, which is a very subjective index of satisfaction. Indeed, the certainty-equivalent is determined by the choice of a utility function, which is specific to each investor. In this section we discuss the second of three broad approaches to model the investor's satisfaction.

To introduce this index of satisfaction, consider a financial institution with a capital of, say, one billion dollars. The financial institution aims at investing its capital in such a way that at a given time horizon the maximum loss does not exceed, say, ten million dollars. Since in a stochastic environment there is no guarantee that the maximum loss will not be exceeded, it is more reasonable to require that the maximum loss is not exceeded within a given confidence margin of, say, ninety-five percent.

We rephrase the above situation in our notation. The investor is the financial institution, which has an initial capital w_T (one billion in the example) at the time the investment decision is made. The institution focuses on the potential loss at the investment horizon, which is the difference between the initial capital w_T and the stochastic capital at the the investment horizon $W_{T+\tau}$. This loss should not exceed the threshold $L_{\max}$ (ten million) with a confidence of at least c (ninety-five percent):

$$\mathbb{P}\left\{w_T - W_{T+\tau} < L_{\max}\right\} \geq c. \tag{5.155}$$

In other words, the investor's objective are net profits, which as in (5.8) depend on the allocation decision

$$\Psi_{\boldsymbol{\alpha}} \equiv W_{T+\tau}\left(\boldsymbol{\alpha}\right) - w_T; \tag{5.156}$$

and from the definition of quantile (1.18) the financial institution manages its investments in such a way that the lower-tail quantile of the net profits, corresponding to a confidence level $1-c$ (five percent), be above the maximum acceptable loss:

$$Q_{\Psi_{\boldsymbol{\alpha}}}\left(1-c\right) \geq -L_{\max}. \tag{5.157}$$

When the required confidence level c is high, the quantile in (5.157) is typically a negative amount, i.e. it represents a loss: the purpose of the financial institution is to make sure that its absolute value does not exceed the maximum loss $L_{\max}$.

The absolute value of the quantile of the objective, when the objective are net profits, is known among practitioners as the *value at risk* (*VaR*) with confidence c of the allocation $\boldsymbol{\alpha}$:

$$\mathrm{VaR}_c\left(\boldsymbol{\alpha}\right) \equiv -Q_{\Psi_{\boldsymbol{\alpha}}}\left(1-c\right). \tag{5.158}$$

The value at risk has become extremely popular among practitioners especially after the Basel Accord, see Crouhy, Galai, and Mark (1998).

More in general the objective $\Psi_{\boldsymbol{\alpha}}$, which depends on the allocation $\boldsymbol{\alpha}$, could be absolute wealth as in (5.3) or relative wealth, as in (5.4), or net profits, as in (5.8), or possibly other specifications. Therefore (5.157) suggests to define a *quantile-based index of satisfaction* of a given allocation $\boldsymbol{\alpha}$ for a generic investor in terms of the quantile of the investor's objective as follows:

$$\boldsymbol{\alpha} \mapsto \mathrm{Q}_c\left(\boldsymbol{\alpha}\right) \equiv Q_{\Psi_{\boldsymbol{\alpha}}}\left(1-c\right), \tag{5.159}$$

where $c \in (0,1)$ is a fixed *confidence level*. This expression is in the form (5.48) and thus it qualifies as a potential index of satisfaction. We see in Figure 5.11 the graphical interpretation of the quantile-based index of satisfaction. We also plot the interpretation of the VaR, which only applies when the investor's objective are the net profits.

5.5.1 Properties

In this section we revisit the properties of a generic index of satisfaction discussed in Section 5.3 to ascertain which are satisfied by the quantile.

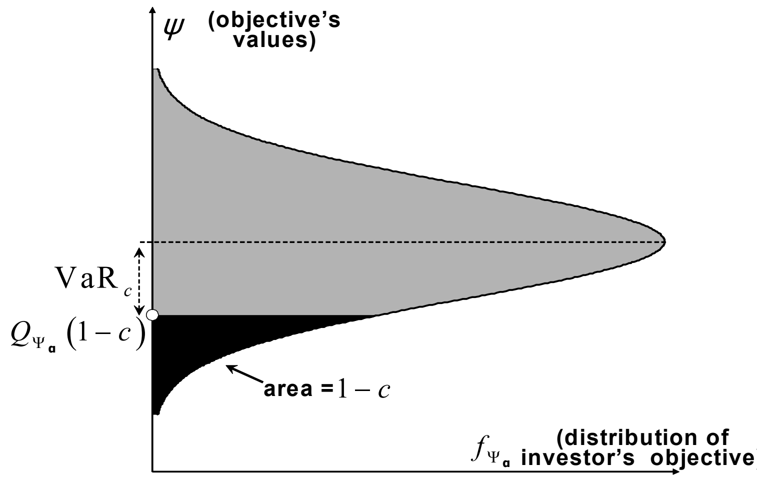

Fig. 5.11. VaR and quantile-based index of satisfaction

- **Money-equivalence**

The quantile of the distribution of the investor's objective has the same dimensions as the objective, which is money. Therefore the quantile-based index of satisfaction (5.159) is a money-equivalent index of satisfaction.

- **Estimability**

The quantile is the inverse of the cumulative distribution function of the objective. Therefore the quantile-based index of satisfaction is defined through a chain such as (5.52) and thus it is estimable:

$$\boldsymbol{\alpha} \mapsto \Psi_{\boldsymbol{\alpha}} \mapsto F_{\Psi_{\boldsymbol{\alpha}}} \mapsto \mathrm{Q}_c(\boldsymbol{\alpha}). \tag{5.160}$$

- **Consistence with stochastic dominance**

The quantile-based index of satisfaction (5.159) is consistent with weak stochastic dominance. The proof is almost tautological. Indeed the quantile satisfies the following relation:

$$Q_{\Psi_{\alpha}}(p) \geq Q_{\Psi_{\beta}}(p) \text{ for all } p \in (0,1) \Rightarrow \mathrm{Q}_c(\boldsymbol{\alpha}) \geq \mathrm{Q}_c(\boldsymbol{\beta}), \tag{5.161}$$

which is the definition of consistence with weak dominance (5.57) in this context.

On the other hand, the quantile is not consistent with second-order dominance. The interested reader can find a counterexample in the context of value at risk in Guthoff, Pfingsten, and Wolf (1997). Since the quantile is not consistent with second-order dominance, (5.61) implies that it cannot be consistent with higher-order dominance.

- **Sensibility**

Since from (5.161) the quantile-based index of satisfaction is consistent with first-order dominance, from (5.61) it is a-fortiori consistent with strong dominance, or order zero dominance. In other words, the quantile-based index of satisfaction satisfies:

$$\Psi_{\boldsymbol{\alpha}} \geq \Psi_{\boldsymbol{\beta}} \text{ in all scenarios } \Rightarrow \mathrm{Q}_c(\boldsymbol{\alpha}) \geq \mathrm{Q}_c(\boldsymbol{\beta}), \tag{5.162}$$

which is the definition of sensibility (5.55) in this context.

- **Constancy**

The quantile-based index of satisfaction (5.159) satisfies the constancy requirement. In other words, we prove in Appendix www.5.4 that for any confidence level $c \in (0,1)$ and any deterministic allocation $\mathbf{b}$ the following relation holds:

$$\Psi_{\mathbf{b}} = \psi_{\mathbf{b}} \Rightarrow \mathrm{Q}_c(\mathbf{b}) = \psi_{\mathbf{b}}, \tag{5.163}$$

which is (5.62) in this context.

- **Positive homogeneity**

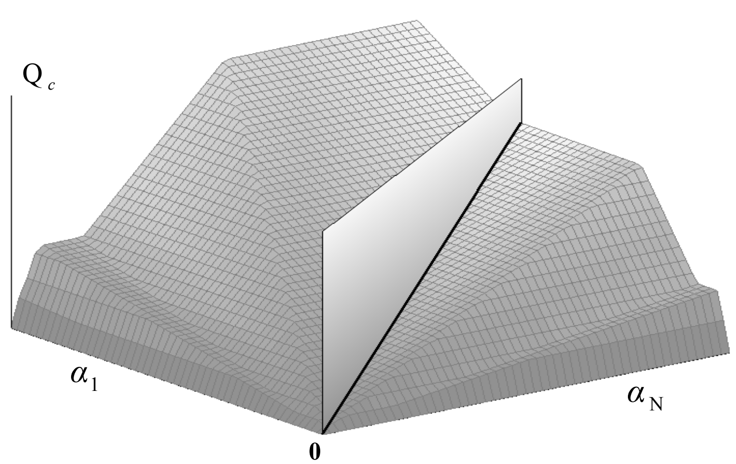

Fig. 5.12. Quantile-based satisfaction index as function of allocation

The quantile-based index of satisfaction is positive homogenous, i.e. it satisfies (5.65), which in this context reads:

$$\mathrm{Q}_c(\lambda\boldsymbol{\alpha}) = \lambda\,\mathrm{Q}_c(\boldsymbol{\alpha}), \quad \text{for all } \lambda \geq 0, \tag{5.164}$$

see Appendix www.5.4 for the proof. In other words, the quantile grows linearly in any radial direction stemming from the origin of the allocation space, see Figure 5.12 and compare with Figure 5.4. Refer to `symmys.com` for details on these figures.

- **Translation invariance**

The quantile-based index of satisfaction is a translation invariant index of satisfaction, i.e. it satisfies (5.72), which in this context reads:

$$\Psi_{\mathbf{b}} \equiv 1 \Rightarrow \mathrm{Q}_c(\boldsymbol{\alpha} + \lambda \mathbf{b}) = \mathrm{Q}_c(\boldsymbol{\alpha}) + \lambda, \tag{5.165}$$

see Appendix www.5.4 for the proof and Figure 5.5 for a geometrical interpretation.

- **Super-/sub- additivity**

The quantile-based index of satisfaction is not super-additive, i.e. in generic markets we have:

$$\mathrm{Q}_c(\boldsymbol{\alpha} + \boldsymbol{\beta}) \ngeq \mathrm{Q}_c(\boldsymbol{\alpha}) + \mathrm{Q}_c(\boldsymbol{\beta}). \tag{5.166}$$

Therefore (5.75) is not satisfied. This is best proved by means of counterexamples, see for instance Artzner, Delbaen, Eber, and Heath (1999) for a counterexample with discrete distributions and Tasche (2002) for a counterexample with continuous distributions, both in the context of value at risk.

Therefore the quantile-based index of satisfaction, and in particular the value at risk, fails to promote diversification. This is the main reason why alternative measures of satisfaction such as the expected shortfall were developed, refer to Section 5.6.

Similarly, the quantile-based index of satisfaction is not sub-additive.

- **Co-monotonic additivity**

The quantile-based index of satisfaction is co-monotonic additive. Indeed, we prove in Appendix www.5.4 that if an allocation $\boldsymbol{\delta}$ gives rise to an objective which is an increasing function of the objective corresponding to another allocation $\boldsymbol{\alpha}$, the satisfaction from the total portfolio is the sum of the satisfactions from the separate allocations:

$$(\boldsymbol{\alpha}, \boldsymbol{\delta}) \text{ co-monotonic } \Rightarrow \mathrm{Q}_c(\boldsymbol{\alpha} + \boldsymbol{\delta}) = \mathrm{Q}_c(\boldsymbol{\alpha}) + \mathrm{Q}_c(\boldsymbol{\delta}). \tag{5.167}$$

This is the definition of co-monotonic additivity (5.80) in this context.

In other words, the combined portfolio is not perceived as providing a diversification effect: the quantile, and in particular the value at risk, are not "fooled" by derivatives.

- **Concavity/convexity**

The quantile-based index of satisfaction (5.159) is neither a concave nor a convex function of the allocation, see Figure 5.12 and refer to `symmys.com` for details on the market behind this figure.

Indeed, the matrix of second-order cross-derivatives of the quantile with respect to the allocation is neither negative definite nor positive definite, see (5.191) and comments thereafter. In other words, quantile-based indices of satisfaction fail to promote diversification, see the discussion on p. 258. This is one of the major critiques that have been directed to the value at risk.

- **Risk aversion/propensity**

The quantile-based index of satisfaction is neither risk averse, nor risk prone, nor risk neural. Indeed, depending on the distribution of the investor's objective and on the level of confidence required, the risk premium (5.85) can assume any sign.

For example, assume that the investor's objective has a Cauchy distribution:

$$\Psi_{\boldsymbol{\alpha}} \sim \mathrm{Ca}\left(\mu, \sigma^2\right). \tag{5.168}$$

Then from (1.82) we obtain:

$$\mathrm{RP}\left(\boldsymbol{\alpha}\right) \equiv \mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\} - Q_{\Psi_{\alpha}}\left(1-c\right) = -\sigma \tan\left(\pi\left(\frac{1}{2}-c\right)\right), \tag{5.169}$$

which can be larger than, equal to, or less than zero depending on the confidence level c.

5.5.2 Explicit dependence on allocation

We recall from (5.10) that the investor's objective, namely absolute wealth, relative wealth, net profits, or other specifications, is a simple linear function of the allocation and the market vector:

$$\Psi_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'\mathbf{M}. \tag{5.170}$$

Therefore the quantile-based index of satisfaction (5.159) depends on the allocation as follows:

$$\boldsymbol{\alpha} \mapsto \mathrm{Q}_c\left(\boldsymbol{\alpha}\right) \equiv Q_{\boldsymbol{\alpha}'\mathbf{M}}\left(1-c\right). \tag{5.171}$$

In this section we tackle the problem of computing explicitly the quantile-based index of satisfaction of an allocation for a given distribution of the market $\mathbf{M}$ and a given choice of the confidence level c.

For example, consider a market which is normally distributed at the investment horizon:

$$\mathbf{P}_{T+\tau} \sim \mathrm{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right). \tag{5.172}$$

Assume that the investor's objective are the net profits as in (5.8). In this case from (5.11) and (5.14) the distribution of the objective reads:

$$\Psi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\mathbf{M} \sim \mathrm{N}\left(\mu_{\boldsymbol{\alpha}}, \sigma^2_{\boldsymbol{\alpha}}\right), \tag{5.173}$$

where

$$\mu_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\left(\boldsymbol{\mu} - \mathbf{P}_T\right), \quad \sigma^2_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}. \tag{5.174}$$

Therefore from (1.70) we have:

$$\mathrm{Q}_c\left(\boldsymbol{\alpha}\right) = \mu_{\boldsymbol{\alpha}} + \sigma_{\boldsymbol{\alpha}}\,\mathrm{erf}^{-1}\left(1 - 2c\right). \tag{5.175}$$

Although the objective is a simple linear function of the allocation and the market, the quantile is in general a complex expression of the allocation. Therefore in general the explicit dependence of the index of satisfaction on the allocation cannot be computed analytically.

There exists a vast literature regarding the computation of the quantile, and in particular of the VaR, using different techniques and under different distributional assumptions for the market, see for instance the list of references at `gloriamundi.org`. We mention here two cases that play a major role in the financial literature: the gamma approximation and the extreme value theory.

Delta-gamma approximation

When the market can be described by the gamma approximation (5.24) and the market invariants are approximately normal, (5.30) yields an approximate expression for the characteristic function of the objective $\Psi_{\boldsymbol{\alpha}}$:

$$\begin{aligned}\phi_{\Psi_{\alpha}}\left(\omega\right) \approx \left|\mathbf{I}_K - i\omega\boldsymbol{\Gamma}_{\boldsymbol{\alpha}}\boldsymbol{\Sigma}\right|^{-\frac{1}{2}} & e^{i\omega\left(\theta_\alpha + \boldsymbol{\Delta}'_\alpha\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{\mu}'\boldsymbol{\Gamma}_\alpha\boldsymbol{\mu}\right)} \\ & e^{-\frac{1}{2}\left[\boldsymbol{\Delta}_\alpha + \boldsymbol{\Gamma}_\alpha\boldsymbol{\mu}\right]'\boldsymbol{\Sigma}\left(\mathbf{I}_K - i\omega\boldsymbol{\Gamma}_\alpha\boldsymbol{\Sigma}\right)^{-1}\left[\boldsymbol{\Delta}_\alpha + \boldsymbol{\Gamma}_\alpha\boldsymbol{\mu}\right]}.\end{aligned} \tag{5.176}$$

The explicit dependence of θ, $\boldsymbol{\Delta}$ and $\boldsymbol{\Gamma}$ on the allocation $\boldsymbol{\alpha}$ is given in (5.26)-(5.28).

From the characteristic function (5.176) we can compute the probability density function of the approximate objective with a numerical inverse Fourier transform as in (1.15) and thus we can compute the quantile Q by solving numerically the following implicit equation:

$$\int_{-\infty}^{\mathrm{Q}} \mathcal{F}^{-1}\left[\phi_{\Psi_\alpha}\right](x)\,dx \equiv 1 - c. \tag{5.177}$$

Nevertheless, this approach does not highlight the explicit dependence of the quantile on the allocation.

To tackle this issue, we can make use of a technique developed by Cornish and Fisher (1937), which expresses the quantile of a generic random variable X in terms of its moments and the quantile of the standard normal distribution:

$$z(p) \equiv \sqrt{2}\,\mathrm{erf}^{-1}(2p-1), \tag{5.178}$$

see (1.70). The *Cornish-Fisher expansion* is an infinite series whose terms can be easily computed up to any order by means of software packages. The first terms read:

$$Q_X(p) = \mathrm{E}\{X\} + \mathrm{Sd}\{X\}\left[z(p) + \frac{1}{6}\left(z^2(p) - 1\right)\mathrm{Sk}\{X\}\right] + \cdots, \tag{5.179}$$

see Kotz, Balakrishnan, and Johnson (1994).

From the characteristic function (5.176) we can recover all the moments of the investor's objective as detailed in Appendix www.5.1. Therefore we can apply the Cornish-Fisher expansion to the investor's objective $\Psi_{\boldsymbol{\alpha}}$ to compute its generic quantile, and thus in turn the quantile-based index of satisfaction (5.159).

In particular, in Appendix www.5.4 we derive the following approximation:

$$\mathrm{Q}_c(\boldsymbol{\alpha}) \approx A_{\boldsymbol{\alpha}} + B_{\boldsymbol{\alpha}} z(1-c) + C_{\boldsymbol{\alpha}} z^2(1-c). \tag{5.180}$$

where

$$\begin{aligned}
A &\equiv \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} - \frac{\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}^3\} - 3\,\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}^2\}\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} + 2\,\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\}^3}{6\left(\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}^2\} - \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\}^2\right)} \\
B &\equiv \sqrt{\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}^2\} - \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\}^2} \\
C &\equiv \frac{\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}^3\} - 3\,\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}^2\}\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} + 2\,\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\}^3}{6\left(\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}^2\} - \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\}^2\right)}.
\end{aligned} \tag{5.181}$$

Refer to Appendix www.5.1 for the explicit dependence on the allocation of the moments that appear in these coefficient.

Approximations of order higher than (5.180) can be obtained similarly.

Extreme value theory

Extreme value theory (EVT) tackles the computation of the quantile when the confidence level in (5.171) is very high, i.e. $c \sim 1$, see Embrechts, Klueppelberg, and Mikosch (1997) and references therein.

Consider the *conditional excess function* of a generic random variable X which is defined as follows:

$$L_{\widetilde{\psi}}(z) \equiv \mathbb{P}\left\{X \leq \widetilde{\psi} - z \mid X \leq \widetilde{\psi}\right\} = \frac{F_X\left(\widetilde{\psi} - z\right)}{F_X\left(\widetilde{\psi}\right)}, \tag{5.182}$$

where F_X denotes the cumulative distribution function of X. The conditional excess function describes the probability that X is less than a generic value below a given threshold, conditioned on X being less than the given threshold.

Consider now the *generalized Pareto cumulative distribution function*, defined for positive z and v and all values of ξ (possibly taking limits for $\xi = 0$) as follows:

$$G_{\xi,v}(z) \equiv 1 - \left(1 + \frac{\xi}{v} z\right)^{-1/\xi}. \tag{5.183}$$

A theorem in Pickands (1975) and Balkema and De Haan (1974) states that under fairly general conditions for very low values of the threshold $\widetilde{\psi}$ in (5.182) there exist suitable values of the parameters ξ and v such that the following approximation holds:

$$1 - L_{\widetilde{\psi}}(z) \approx G_{\xi,v}(z). \tag{5.184}$$

Substituting (5.183) in (5.184) and applying this result to the investor's objective $X \equiv \Psi_{\boldsymbol{\alpha}}$ we obtain an approximation for the cumulative distribution function of the objective for very low values of its range:

$$F_{\Psi_{\alpha}}(\psi) \approx F_{\Psi_{\alpha}}\left(\widetilde{\psi}\right)\left(1 + \frac{\xi(\boldsymbol{\alpha})}{v(\boldsymbol{\alpha})}\left(\widetilde{\psi} - \psi\right)\right)^{-1/\xi}. \tag{5.185}$$

Inverting this relation we obtain the approximate expression for the quantile-based index of satisfaction:

$$\mathrm{Q}_c(\boldsymbol{\alpha}) \approx \widetilde{\psi} + \frac{v(\boldsymbol{\alpha})}{\xi(\boldsymbol{\alpha})}\left[1 - \left(\frac{1-c}{F_{\Psi_{\alpha}}\left(\widetilde{\psi}\right)}\right)^{-\xi(\boldsymbol{\alpha})}\right]. \tag{5.186}$$

Nevertheless, the applicability of this formula in this context is limited because the dependence on the allocation $\boldsymbol{\alpha}$ of the parameters v and ξ and of the threshold cdf $F_{\Psi_{\alpha}}\left(\widetilde{\psi}\right)$ is non-trivial.

5.5.3 Sensitivity analysis

Suppose that the investor has already chosen an allocation $\boldsymbol{\alpha}$ which yields a level of satisfaction $\mathrm{Q}_c(\boldsymbol{\alpha})$ and that he is interested in rebalancing his portfolio marginally by means of a small change $\delta\boldsymbol{\alpha}$ in the current allocation. In this case a local analysis in terms of a Taylor expansion is useful:

$$\begin{aligned}\mathrm{Q}_c(\boldsymbol{\alpha} + \delta\boldsymbol{\alpha}) \approx \mathrm{Q}_c(\boldsymbol{\alpha}) &+ \delta\boldsymbol{\alpha}' \frac{\partial\,\mathrm{Q}_c(\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}} \\ &+ \frac{1}{2}\delta\boldsymbol{\alpha}' \frac{\partial^2\,\mathrm{Q}_c(\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\alpha}'}\delta\boldsymbol{\alpha}.\end{aligned} \tag{5.187}$$

We prove in Appendix www.5.4 that the first-order derivatives read:

$$\frac{\partial\,\mathrm{Q}_c(\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}} = \mathrm{E}\left\{\mathbf{M} | \boldsymbol{\alpha}'\mathbf{M} = \mathrm{Q}_c(\boldsymbol{\alpha})\right\}, \tag{5.188}$$

where $\mathbf{M}$ is the random vector (5.170) that represents the market, see Hallerbach (2003), Gourieroux, Laurent, and Scaillet (2000), Tasche (2002). The investor will focus on the entries of the vector (5.188) that display a large absolute value.

For example, in the case of normal markets from (5.175) we obtain directly the first-order derivatives:

$$\frac{\partial \,\mathrm{Q}_c(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} = \boldsymbol{\mu} - \mathbf{p}_T + \frac{\boldsymbol{\Sigma}\boldsymbol{\alpha}}{\sqrt{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}}\sqrt{2}\,\mathrm{erf}^{-1}(1-2c)\,. \tag{5.189}$$

The quantile-based index of satisfaction is positive homogeneous, see (5.164). Therefore it can be expressed in terms of the contribution to satisfaction from each security by means of the Euler decomposition (5.67), which in this context reads:

$$\mathrm{Q}_c(\boldsymbol{\alpha}) = \sum_{n=1}^{N} \alpha_n \,\mathrm{E}\{M_n | \boldsymbol{\alpha}'\mathbf{M} = \mathrm{Q}_c(\boldsymbol{\alpha})\}\,. \tag{5.190}$$

The contribution to satisfaction from each security in turn factors into the product of the amount of that security times the marginal contribution to satisfaction of that security. The marginal contribution to satisfaction, i.e. the conditional expectation of the market vector (5.188), is insensitive to a rescaling of the portfolio, although it depends on the allocation.

For example, consider the case of normal markets. Left-multiplying the marginal contributions to satisfaction (5.189) by the amount of each security $\boldsymbol{\alpha}'$ we obtain the quantile-based index of satisfaction (5.175). Also, a rescaling $\boldsymbol{\alpha} \mapsto \lambda\boldsymbol{\alpha}$ does not affect (5.189).

The study of the second-order cross-derivatives provides insight in the local convexity/concavity of the certainty-equivalent. In Appendix www.5.4 we adapt from Gourieroux, Laurent, and Scaillet (2000) to prove the following result:

$$\begin{aligned}\frac{\partial^2 \,\mathrm{Q}_c(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}'\partial \boldsymbol{\alpha}} = &- \left.\frac{\partial \ln f_{\Psi_{\boldsymbol{\alpha}}}(\psi)}{\partial \psi}\right|_{\psi=\mathrm{Q}_c(\boldsymbol{\alpha})} \mathrm{Cov}\{\mathbf{M}|\Psi_{\boldsymbol{\alpha}} = \mathrm{Q}_c(\boldsymbol{\alpha})\} \\ &- \left.\frac{\partial \,\mathrm{Cov}\{\mathbf{M}|\Psi_{\boldsymbol{\alpha}} = \psi\}}{\partial \psi}\right|_{\psi=\mathrm{Q}_c(\boldsymbol{\alpha})},\end{aligned} \tag{5.191}$$

where $f_{\Psi_{\boldsymbol{\alpha}}}$ is the probability density function of the investor's objective $\Psi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\mathbf{M}$.

For example, consider the case of normal markets. By direct derivation of (5.189) we obtain:

$$\frac{\partial^2 \mathrm{Q}_c(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}' \partial \boldsymbol{\alpha}} = \boldsymbol{\Sigma} \left(\mathbf{I}_N - \frac{\boldsymbol{\alpha}\boldsymbol{\alpha}'\boldsymbol{\Sigma}}{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}} \right) \frac{\sqrt{2}\,\mathrm{erf}^{-1}(1-2c)}{\sqrt{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}}. \tag{5.192}$$

In the normal case the second term in (5.191) vanishes because the conditional covariance in does not depend on the value on which it is conditioned.

In Appendix www.5.4 we prove that the matrix (5.192) is negative definite for high confidence levels, namely $c > 0.5$, and positive definite for low confidence levels, namely $c < 0.5$. Therefore for high confidence levels the quantile is concave and for low confidence levels the quantile is convex.

In general the first term in (5.191) is fairly easy to analyze: the conditional covariance is always positive definite and the elasticity of the marginal density is typically positive in the lower tail, which corresponds to a high level of confidence $c \sim 1$ in the quantile. Therefore for high levels of confidence the first term is negative definite and tends to make the quantile a concave index of satisfaction. Similarly, the elasticity of the marginal density is typically negative in the upper tail, which corresponds to a low level of confidence $c \sim 0$ in the quantile. Therefore for low levels of confidence the first term is positive definite and tends to make the quantile a convex index of satisfaction.

On the other hand, the sign of the second term in (5.191) is not determined. Therefore in general the quantile-based index of satisfaction is neither convex nor concave, see Figure 5.12.

5.6 Coherent indices (expected shortfall)

In Section 5.4 we discussed a first class of indices of satisfaction based on expected utility, namely the certainty-equivalent, and in Section 5.5 we introduced a second class of indices, namely the quantile-based indices of satisfaction. In this section, following the recent literature on measures of risk, we discuss a third approach to model the investor's satisfaction, namely coherent indices of satisfaction, a class of indices of satisfaction that are defined directly in terms of the properties that they are supposed to feature, see Artzner, Delbaen, Eber, and Heath (1997) and Artzner, Delbaen, Eber, and Heath (1999).

Such indices originated from the critiques directed to the value at risk for not promoting diversification. Indeed, the quantile-based index of satisfaction is not a concave function of allocation, see Figure 5.12. Therefore it fails to promote diversification. Although diversification is not necessarily a requirement for the portfolio of a private investor, it is an important requirement in the investment policy of a financial institution.

5.6.1 Properties

In this section we introduce the broad class of *coherent indices of satisfaction*, namely functions that with a generic allocation $\boldsymbol{\alpha}$ associate a level of satisfaction $\mathrm{Coh}(\boldsymbol{\alpha})$ in such a way to satisfy the properties discussed below. Since these indices are defined axiomatically in terms of their properties, we revisit the properties of a generic index of satisfaction introduced in Section 5.3 highlighting which of them define coherent indices of satisfaction and which of them are consequences of the definitions.

Consider an investor with a given objective Ψ such as absolute wealth, as in (5.3), or relative wealth, as in (5.4), or net profits, as in (5.8), or possibly other specifications. As usual, we denote as $\Psi_{\boldsymbol{\alpha}}$ the the objective relative to a generic allocation $\boldsymbol{\alpha}$, see (5.10).

- **Sensibility (definition)**

A coherent index of satisfaction must be sensible, i.e. consistent with strong dominance:

$$\Psi_{\boldsymbol{\alpha}} \geq \Psi_{\boldsymbol{\beta}} \text{ in all scenarios } \Rightarrow \mathrm{Coh}(\boldsymbol{\alpha}) \geq \mathrm{Coh}(\boldsymbol{\beta}), \tag{5.193}$$

which is (5.55) in this context.

- **Positive homogeneity (definition)**

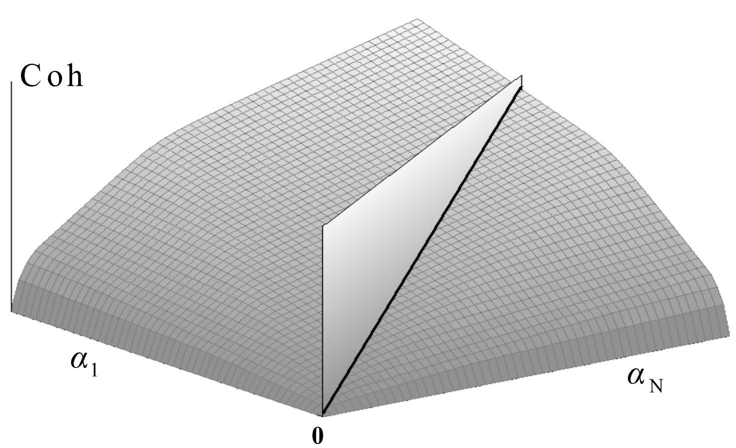

Fig. 5.13. Coherent satisfaction index as function of allocation

A coherent index of satisfaction must be positive homogeneous:

$$\operatorname{Coh}(\lambda\boldsymbol{\alpha}) = \lambda \operatorname{Coh}(\boldsymbol{\alpha}), \quad \text{for all } \lambda \geq 0, \tag{5.194}$$

which is (5.65) in this context. In other words, the coherent index grows linearly in any radial direction stemming from the origin of the allocation space, see Figure 5.13, which refers to the same market as Figure 5.12. Refer to `symmys.com` for details on these figures.

Notice that the requirement of positive homogeneity rules out the certainty-equivalent as a coherent index of satisfaction, unless the utility function is of the power class.

- **Translation invariance (definition)**

A coherent index of satisfaction must be translation invariant:

$$\Psi_{\mathbf{b}} \equiv 1 \Rightarrow \operatorname{Coh}(\boldsymbol{\alpha} + \lambda\mathbf{b}) = \operatorname{Coh}(\boldsymbol{\alpha}) + \lambda, \tag{5.195}$$

which is (5.72) in this context, see Figure 5.5 for a geometrical interpretation.

Notice that the requirement of translation invariance rules out the certainty-equivalent as a coherent index of satisfaction, unless the utility function is of the exponential class.

The only intersection between the exponential class and the power class is the trivial linear utility function, in which case the certainty-equivalent becomes the expected value of the investor's objective, which is our first example of coherent index of satisfaction:

$$\operatorname{Coh}(\boldsymbol{\alpha}) \equiv \operatorname{CE}(\boldsymbol{\alpha}) \equiv \operatorname{E}\{\Psi_{\boldsymbol{\alpha}}\}. \tag{5.196}$$

- **Super-additivity (definition)**

A coherent index of satisfaction must be super-additive:

$$\operatorname{Coh}(\boldsymbol{\alpha} + \boldsymbol{\beta}) \geq \operatorname{Coh}(\boldsymbol{\alpha}) + \operatorname{Coh}(\boldsymbol{\beta}), \tag{5.197}$$

which is (5.75) in this context. Notice that this requirement rules out the quantile-based index of satisfaction, see (5.166). Also, this requirement rules out the certainty-equivalent as a coherent index of satisfaction, except for the trivial case of a linear utility function (5.196).

A notable class of coherent indices of satisfaction are the one-sided moments, see Fischer (2003):

$$\operatorname{Coh}(\boldsymbol{\alpha}) \equiv \operatorname{E}\{\Psi_{\boldsymbol{\alpha}}\} - \gamma \left\| \min\left(0, \Psi_{\boldsymbol{\alpha}} - \operatorname{E}\{\Psi_{\boldsymbol{\alpha}}\}\right) \right\|_{\mathbf{M};p}. \tag{5.198}$$

In this expression $\gamma \geq 0$ and $\|\cdot\|_{\mathbf{M};p}$ is the market-based expectation norm $(B.57)$, which in terms of the market probability density function $f_{\mathbf{M}}$ reads:

$$\|g\|_{\mathbf{M};p} \equiv \left(\int |g(\mathbf{m})|^p f_{\mathbf{M}}(\mathbf{m})\, d\mathbf{m} \right)^{\frac{1}{p}} . \tag{5.199}$$

Notice that (5.198) is defined in terms of the distribution of the objective $\Psi_{\boldsymbol{\alpha}}$ and as such it is law invariant, or estimable, see also below.

The trivial case $\gamma \equiv 0$ recovers the expected value (5.196) as a coherent index of satisfaction.

The specific case $\gamma \equiv 1$ and $p \equiv 2$ gives rise to the *mean/semistandard deviation*, which practitioners use extensively.

The above four defining properties imply other features.

- **Money-equivalence (consequence of definition)**

The joint assumptions of positive homogeneity and translation invariance imply that a coherent index of satisfaction is naturally measured in terms of money.

- **Concavity (consequence of definition)**

The joint assumptions of positive homogeneity and super-additivity imply that a coherent index of satisfaction is concave, i.e. for all $\lambda \in [0,1]$ the following holds true:

$$\operatorname{Coh}(\lambda\boldsymbol{\alpha} + (1-\lambda)\boldsymbol{\beta}) \geq \lambda \operatorname{Coh}(\boldsymbol{\alpha}) + (1-\lambda)\operatorname{Coh}(\boldsymbol{\beta}), \tag{5.200}$$

which is (5.81) in this context.

In other words, coherent indices of satisfaction promote diversification by construction, see Figure 5.13. We recall that this is not the case for quantile-based indices of satisfaction such as the value at risk, see Figure 5.12, which refers to the same market as Figure 5.13. Refer to `symmys.com` for details on these figures.

The above properties of the coherent indices of satisfaction cover many but not all the potential features of a generic index of satisfaction discussed in Section 5.3.

Adding a few more intuitive requirements, Acerbi (2002) introduced the sub-class of coherent indices known as *spectral indices of satisfaction*[5]. Spectral indices of satisfaction are functions that with a generic allocation $\boldsymbol{\alpha}$ associate a level of satisfaction $\operatorname{Spc}(\boldsymbol{\alpha})$ which satisfies the properties of coherent indices of satisfaction and the additional properties discussed below. Indeed, spectral indices of satisfaction cover all the potential features of a generic index of satisfaction discussed in Section 5.3. Again, we distinguish between truly new defining features and simple consequences of the definitions.

- **Estimability (definition)**

[5] The reason for this terminology will become apparent in Section 5.6.2.

A spectral index of satisfaction is fully determined by the distribution of the investor's objective $\Psi_{\boldsymbol{\alpha}}$, as represented by either its probability density function, cumulative distribution function, or characteristic function. In other words, a spectral index of satisfaction is defined in terms of the following chain:

$$\boldsymbol{\alpha} \mapsto \Psi_{\boldsymbol{\alpha}} \mapsto \left(f_{\Psi_{\boldsymbol{\alpha}}}, F_{\Psi_{\boldsymbol{\alpha}}}, \phi_{\Psi_{\boldsymbol{\alpha}}}\right) \mapsto \operatorname{Spc}(\boldsymbol{\alpha}), \tag{5.201}$$

which is (5.52) in this context.

- **Co-monotonic additivity (definition)**

A spectral index of satisfaction is co-monotonic additive:

$$(\boldsymbol{\alpha}, \boldsymbol{\delta}) \text{ co-monotonic } \Rightarrow \operatorname{Spc}(\boldsymbol{\alpha}+\boldsymbol{\delta}) = \operatorname{Spc}(\boldsymbol{\alpha}) + \operatorname{Spc}(\boldsymbol{\delta}), \tag{5.202}$$

which is (5.80) in this context. In other words, like (non-coherent) quantile-based indices of satisfaction such as the value at risk, the spectral indices of satisfaction are not "fooled" by derivatives: whenever the objective relative to one allocation is an increasing function of the objective stemming from another allocation, the combined portfolio is not perceived as providing a diversification effect.

For example the one-sided moments (5.198) are estimable, but not co-monotonic additive. Therefore they give rise to coherent indices of satisfaction, but not to spectral indices of satisfaction. On the other hand, consider the expected value of the objective:

$$\operatorname{Spc}(\boldsymbol{\alpha}) \equiv \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\}. \tag{5.203}$$

From (5.77) the expected value is additive and thus in particular it is co-monotonic additive. Furthermore, from (5.54) it is estimable and from (5.196) it is coherent. Therefore the expected value of the investor's objective is a spectral index of satisfaction.

The above defining properties of spectral indices of satisfaction imply the following features.

- **Consistence with weak stochastic dominance (consequence of definition)**

From (5.59) sensibility and estimability imply consistence with weak stochastic dominance. Therefore the spectral indices of satisfaction are consistent with weak stochastic dominance:

$$Q_{\Psi_{\alpha}}(p) \geq Q_{\Psi_{\beta}}(p) \text{ for all } p \in (0,1) \Rightarrow \operatorname{Spc}(\boldsymbol{\alpha}) \geq \operatorname{Spc}(\boldsymbol{\beta}), \tag{5.204}$$

which is (5.57) in this context.

- **Constancy (consequence of definition)**

Translation invariance and homogeneity imply constancy, see Appendix www.5.2. Therefore the spectral indices of satisfaction satisfy:

$$\Psi_{\mathbf{b}} \equiv \psi_{\mathbf{b}} \Rightarrow \operatorname{Spc}(\mathbf{b}) = \psi_{\mathbf{b}}, \tag{5.205}$$

which is (5.62) in this context.

- **Risk aversion (consequence of definition)**

Spectral indices of satisfaction are risk averse. Indeed, we prove in Appendix www.5.5 that the risk premium of an allocation associated with a spectral index of satisfaction is positive: this is the definition of risk aversion (5.86).

5.6.2 Building coherent indices

In this section we discuss how to build coherent indices of satisfaction. More precisely, we focus on spectral indices. We proceed as in the case of the certainty-equivalent, see Section 5.4.2. In other words, we specify a basis for the spectral indices of satisfaction and then we obtain all possible indices as weighted averages of the basis.

To define a basis, we start from the only example of spectral index introduced so far, namely the expected value of the investor's objective (5.203). In order to build other elements for the basis of the spectral indices we generalize the expected value, which, with a change of variables, we can express as the average of all the quantiles:

$$\begin{aligned} \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} &= \int_{\mathbb{R}} \psi f_{\Psi_{\alpha}}(\psi)\, d\psi \\ &= \int_0^1 Q_{\Psi_{\alpha}}(s)\, ds. \end{aligned} \tag{5.206}$$

One the one hand, it is important to start with the quantiles. Indeed, the quantile is co-monotonic additive, see (5.167), and estimable, see (5.160). These are exactly the two new features required of spectral indices of satisfaction.

On the other hand, it is important to suitably average a given range of quantiles, because the quantile per se is not super-additive, see (5.166). Super-additivity is a key feature of coherent measures of risk, and thus in particular it is a key feature of spectral indices of satisfaction.

Therefore, we define new indices of satisfaction as averages of quantiles. To make the ensuing index of satisfaction more conservative, we average the quantiles over the worst scenarios. This way we obtain the definition of *expected shortfall*[6] (*ES*):

[6] The name expected shortfall applies specifically to the case where the investor's objective are net profits, much like the value at risk in the context of quantile-based indices of satisfaction. For simplicity, we extend the terminology to a generic objective, such as absolute or relative wealth.

$$\mathrm{ES}_c(\boldsymbol{\alpha}) \equiv \frac{1}{1-c} \int_0^{1-c} Q_{\Psi_\alpha}(s)\, ds, \tag{5.207}$$

where $c \in [0,1]$ is a fixed *confidence level*. The expected shortfall is indeed a generalization of the expected value, which represents the specific case $c \equiv 0$.

Notice that the expected shortfall is the expected value of the investor's objective, conditioned on the realization of the objective being less than the quantile-based index of satisfaction (5.159). Therefore the expected shortfall is the *tail conditional expectation* (*TCE*), also known as *conditional value at risk*[7] (*CVaR*):

$$\begin{aligned}\mathrm{ES}_c(\boldsymbol{\alpha}) &= \mathrm{TCE}_c(\boldsymbol{\alpha}) = \mathrm{CVaR}_c(\boldsymbol{\alpha}) \\ &\equiv \mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}} | \Psi_{\boldsymbol{\alpha}} \leq \mathrm{Q}_c(\boldsymbol{\alpha})\right\}.\end{aligned} \tag{5.208}$$

We now verify that, just like the expected value, the expected shortfall satisfies the defining properties of a spectral index of satisfaction.

- **Sensibility**

The expected shortfall is sensible:

$$\Psi_{\boldsymbol{\alpha}} \geq \Psi_{\boldsymbol{\beta}} \text{ in all scenarios } \Rightarrow \mathrm{ES}_c(\boldsymbol{\alpha}) \geq \mathrm{ES}_c(\boldsymbol{\beta}). \tag{5.209}$$

Indeed, from (5.57) and (5.207) it is consistent with first-order dominance, and thus from (5.61) it is consistent with strong dominance.

- **Positive homogeneity**

The expected shortfall is positive homogeneous:

$$\mathrm{ES}_c(\lambda\boldsymbol{\alpha}) = \lambda\, \mathrm{ES}_c(\boldsymbol{\alpha}), \qquad \lambda \geq 0, \tag{5.210}$$

see Figure 5.13. This follows from the linearity of the integral in the definition of expected shortfall and the positive homogeneity of the quantile proved in Appendix www.5.4.

- **Translation invariance**

The expected shortfall is translation invariant:

$$\mathrm{ES}_c(\boldsymbol{\alpha} + \lambda\mathbf{b}) = \mathrm{ES}_c(\boldsymbol{\alpha}) + \lambda, \tag{5.211}$$

see Figure 5.5 for a geometrical interpretation. This follows from the linearity of the integral in the definition of expected shortfall and the translation invariance of the quantile proved in Appendix www.5.4.

- **Super-additivity**

[7] We assume that the probability density function of the objective f_{Ψ_α} is smooth, otherwise this is not true, see Acerbi and Tasche (2002) for a counterexample.

The expected shortfall is super-additive:

$$\mathrm{ES}_c(\boldsymbol{\alpha}+\boldsymbol{\beta}) \geq \mathrm{ES}_c(\boldsymbol{\alpha}) + \mathrm{ES}_c(\boldsymbol{\beta}). \tag{5.212}$$

For the proof of super-additivity we refer the interested reader to Acerbi and Tasche (2002).

We stress that in particular the joint assumptions of positive homogeneity and super-additivity imply that, unlike the value at risk, the expected shortfall is concave. In other words, for all $\lambda \in [0,1]$ the following holds true:

$$\mathrm{ES}_c(\lambda\boldsymbol{\alpha}+(1-\lambda)\boldsymbol{\beta}) \geq \lambda\,\mathrm{ES}_c(\boldsymbol{\alpha}) + (1-\lambda)\,\mathrm{ES}_c(\boldsymbol{\beta}). \tag{5.213}$$

We see this in Figure 5.13, which refers to a market of two securities. Compare also with Figure 5.12, which refers to the value at risk in the same market. See `symmys.com` for details on these figures.

- **Estimability**

The expected shortfall is estimable, since the quantile in the definition (5.207) of expected shortfall is the inverse of the cumulative distribution function F_{Ψ_α} of the investor's objective. Therefore a chain-definition such as (5.201) applies:

$$\boldsymbol{\alpha} \mapsto \Psi_{\boldsymbol{\alpha}} \mapsto F_{\Psi_\alpha} \mapsto \mathrm{ES}_c(\boldsymbol{\alpha}). \tag{5.214}$$

- **Co-monotonic additivity**

The expected shortfall is co-monotonic additive:

$$(\boldsymbol{\alpha},\boldsymbol{\delta}) \text{ co-monotonic } \Rightarrow \mathrm{ES}_c(\boldsymbol{\alpha}+\boldsymbol{\delta}) = \mathrm{ES}_c(\boldsymbol{\alpha}) + \mathrm{ES}_c(\boldsymbol{\delta}). \tag{5.215}$$

This follows from the linearity of the integral in the definition (5.207) of expected shortfall and the co-monotonic additivity of the quantile proved in Appendix www.5.4.

Since it satisfies the defining properties of spectral indices of satisfaction, the expected shortfall belongs to this class for any value of the confidence level c. Therefore we choose the expected shortfall as a basis to generate other spectral indices of satisfaction. In other words, we consider all weighted averages of the expected shortfall, in a way completely similar to the construction of utility functions (5.125). As we prove in Appendix www.5.5, this way we obtain the following class of spectral indices of satisfaction:

$$\mathrm{Spc}_\varphi(\boldsymbol{\alpha}) \equiv \int_0^1 \varphi(p)\,Q_{\Psi_\alpha}(p)\,dp, \tag{5.216}$$

where the *spectrum* φ is a function that satisfies:

$$\varphi \text{ decreasing}, \quad \varphi(1) \equiv 0, \quad \int_0^1 \varphi(p)\,dp \equiv 1. \tag{5.217}$$

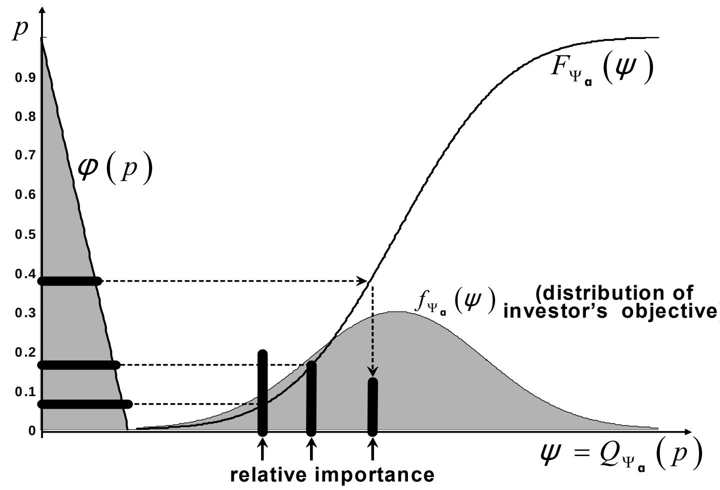

Fig. 5.14. Spectral indices of satisfaction emphasize adverse scenarios

In other words, spectral indices of satisfaction give relatively speaking more importance to the unwelcome scenarios in which the investor's objective is low, see Figure 5.14. This feature makes spectral indices of satisfaction risk averse.

As it turns out, the class (5.216) is exhaustive, as any spectral index of satisfaction can be expressed this way for a suitable choice of the spectrum φ, see Kusuoka (2001) or Tasche (2002). This clarifies why this class of indices of satisfaction is called "spectral".

For example, the expected shortfall can be represented in the form (5.216) by the following spectrum:

$$\varphi_{\mathrm{ES}_c}(p) \equiv \frac{H^{(c-1)}(-p)}{1-c}, \tag{5.218}$$

where $H^{(x)}$ is the Heaviside step function $(B.74)$. It is easy to check that this spectrum satisfies the requirements (5.217).

The requirements (5.217) on the spectrum are essential to obtain a coherent index.

For example, also the quantile-based index of satisfaction can be represented in the form (5.216) by the following spectrum:

$$\varphi_{\mathrm{Q}_c} \equiv \delta^{(1-c)}, \tag{5.219}$$

where δ is the Dirac delta ($B.16$). Nevertheless, the Dirac-delta is (the limit of) a bell-shaped function ($B.18$). Therefore, the spectrum is not decreasing as prescribed in (5.217). Indeed, quantile-based indices of satisfaction, and in particular the value at risk, are not coherent.

We remark that with a change of variable in (5.216) any spectral index of satisfaction can be written as follows:

$$\mathrm{Spc}_{\varphi}(\boldsymbol{\alpha}) = \int_{-\infty}^{+\infty} \psi\varphi\left(F_{\Psi_{\boldsymbol{\alpha}}}(\psi)\right) f_{\Psi_{\boldsymbol{\alpha}}}(\psi)\, d\psi. \tag{5.220}$$

This is the expression of the expected utility (5.90), where the utility function is defined as follows:

$$u(\psi) \equiv \psi\varphi\left(F_{\Psi_{\boldsymbol{\alpha}}}(\psi)\right). \tag{5.221}$$

Therefore, as in (5.102) we can interpret the utility function as a subjective cumulative distribution function that reflects the investor's a-priori view on the outcome of his investment. Nevertheless, in this case the utility function depends not only on the investor's attitude toward risk, i.e. the spectral function φ, but also on the market and the allocation decision through the cumulative distribution function $F_{\Psi_{\boldsymbol{\alpha}}}$.

5.6.3 Explicit dependence on allocation

We recall from (5.10) that the investor's objective, namely absolute wealth, relative wealth, net profits, or other specifications, is a simple linear function of the allocation and the market vector:

$$\Psi_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'\mathbf{M}. \tag{5.222}$$

Therefore the spectral indices of satisfaction (5.216) depends on the allocation as follows:

$$\boldsymbol{\alpha} \mapsto \mathrm{Spc}_{\varphi}(\boldsymbol{\alpha}) \equiv \int_{0}^{1} \varphi(p)\, Q_{\boldsymbol{\alpha}'\mathbf{M}}(p)\, dp. \tag{5.223}$$

Notice that the quantile depends on the allocation and on the distribution of the market, whereas the spectrum does not.

In this section we tackle the problem of computing explicitly spectral indices of satisfaction for a given distribution of the market $\mathbf{M}$ and a given choice of the spectrum φ.

For example, consider a market that at the investment horizon is normally distributed:

$$\mathbf{P}_{T+\tau} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{5.224}$$

Assume that the investor's objective are the net profits as in (5.8). In this case from (5.11) and (5.14) the distribution of the objective reads:

$$\Psi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\mathbf{M} \sim \mathrm{N}\left(\mu_{\boldsymbol{\alpha}}, \sigma^2_{\boldsymbol{\alpha}}\right), \tag{5.225}$$

where

$$\mu_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\left(\boldsymbol{\mu} - \mathbf{P}_T\right), \qquad \sigma^2_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}. \tag{5.226}$$

From (1.70) the quantile of the investor's objective reads:

$$Q_{\boldsymbol{\alpha}'\mathbf{M}}(p) = \mu_{\boldsymbol{\alpha}} + \sigma_{\boldsymbol{\alpha}} z(p), \tag{5.227}$$

where z is the quantile of the standard normal distribution:

$$z(p) \equiv \sqrt{2}\,\mathrm{erf}^{-1}(2p-1). \tag{5.228}$$

Therefore in this market the generic spectral index of satisfaction reads:

$$\mathrm{Spc}_{\varphi}(\boldsymbol{\alpha}) = \mu_{\boldsymbol{\alpha}} + \sigma_{\boldsymbol{\alpha}}\mathcal{I}[\varphi z], \tag{5.229}$$

where $\mathcal{I}$ denotes the integration operator $(B.27)$. Notice that the integral does not depend on the allocation.

In generic markets the quantile of the objective in (5.223) is a complex expression of the allocation. Therefore in general the explicit dependence of a spectral index of satisfaction on allocation cannot be computed analytically.

Nevertheless, as for the quantile-based indices of satisfaction, we mention here two special quite general cases where we can compute approximate expressions for the expected shortfall: the delta-gamma approximation and extreme value theory.

Delta-gamma approximation

When the market can be described by the gamma approximation (5.24) and the market invariants are approximately normal, (5.30) yields an approximate expression for the characteristic function of the objective $\Psi_{\boldsymbol{\alpha}}$:

$$\begin{aligned}\phi_{\Psi_{\alpha}}(\omega) \approx \left|\mathbf{I}_K - i\omega\boldsymbol{\Gamma}_{\boldsymbol{\alpha}}\boldsymbol{\Sigma}\right|^{-\frac{1}{2}} & e^{i\omega\left(\theta_\alpha + \boldsymbol{\Delta}'_\alpha\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{\mu}'\boldsymbol{\Gamma}_\alpha\boldsymbol{\mu}\right)} \\ & e^{-\frac{1}{2}[\boldsymbol{\Delta}_\alpha + \boldsymbol{\Gamma}_\alpha\boldsymbol{\mu}]'\boldsymbol{\Sigma}(\mathbf{I}_K - i\omega\boldsymbol{\Gamma}_\alpha\boldsymbol{\Sigma})^{-1}[\boldsymbol{\Delta}_\alpha + \boldsymbol{\Gamma}_\alpha\boldsymbol{\mu}]}.\end{aligned} \tag{5.230}$$

The explicit dependence of θ, $\boldsymbol{\Delta}$ and $\boldsymbol{\Gamma}$ on the allocation $\boldsymbol{\alpha}$ is given in (5.26)-(5.28).

From the characteristic function we can compute the probability density function of the approximate objective with a numerical inverse Fourier transform as in (1.15) and then we can compute the quantile by solving numerically the following implicit equation:

$$\int_{-\infty}^{Q} \mathcal{F}^{-1}\left[\phi_{\Psi_\alpha}\right](x)\,dx \equiv p. \tag{5.231}$$

Finally, a third numerical integration yields the spectral index of satisfaction (5.223). Nevertheless, this approach is computationally intensive and unstable and does not highlight the explicit dependence of the spectral index on allocation on the allocation.

To tackle this issue, we can use the Cornish-Fisher expansion (5.179). In Appendix www.5.5 we show that an approximate expression of the spectral index of satisfaction in terms of the integration operator $(B.27)$ reads:

$$\operatorname{Spc}_{\varphi}(\boldsymbol{\alpha}) \approx A_{\boldsymbol{\alpha}} + B_{\boldsymbol{\alpha}} \mathcal{I}[\varphi z] + C_{\boldsymbol{\alpha}} \mathcal{I}\left[\varphi z^2\right], \tag{5.232}$$

where the coefficients A, B, C are defined in (5.181). Notice that the integrals in this expression do not depend on the allocation: therefore they can be evaluated numerically once and for all. Higher-order approximations can be obtained similarly.

Extreme value theory

Extreme value theory does not apply to the computation of spectral measures of satisfaction in general. Nevertheless, it does apply to the computation of the most notable among the spectral measures of satisfaction, namely the expected shortfall, when the confidence level in (5.207) is very high, i.e. $c \sim 1$.

As we show in Appendix www.5.5 there exist functions of the allocation $v(\boldsymbol{\alpha})$ and $\xi(\boldsymbol{\alpha})$ such that the expected shortfall can be approximated as follows:

$$\operatorname{ES}_c(\boldsymbol{\alpha}) \approx \operatorname{Q}_c(\boldsymbol{\alpha}) - \frac{v(\boldsymbol{\alpha})}{1-\xi(\boldsymbol{\alpha})}, \tag{5.233}$$

where the parameters v and ξ and the extreme quantile $\operatorname{Q}_c(\boldsymbol{\alpha})$ are as in (5.186). Nevertheless, the applicability of this formula in this context is limited because the explicit dependence on the allocation of the parameters v and ξ and of the quantile-based index of satisfaction Q_c is non-trivial.

5.6.4 Sensitivity analysis

Suppose that the investor has already chosen an allocation $\boldsymbol{\alpha}$ which yields a level of satisfaction $\operatorname{Spc}_{\varphi}(\boldsymbol{\alpha})$ and that he is interested in rebalancing his portfolio marginally by means of a small change $\delta\boldsymbol{\alpha}$ in the current allocation. In this case a local analysis in terms of a Taylor expansion is useful:

$$\begin{aligned}\operatorname{Spc}_{\varphi}(\boldsymbol{\alpha}+\delta\boldsymbol{\alpha}) \approx & \operatorname{Spc}_{\varphi}(\boldsymbol{\alpha}) + \delta\boldsymbol{\alpha}' \frac{\partial \operatorname{Spc}_{\varphi}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} \\ & + \frac{1}{2}\delta\boldsymbol{\alpha}' \frac{\partial^2 \operatorname{Spc}_{\varphi}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\alpha}'} \delta\boldsymbol{\alpha}.\end{aligned} \tag{5.234}$$

In Appendix www.5.5 we show that the first-order derivatives read:

$$\frac{\partial \operatorname{Spc}_{\varphi}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} = -\int_0^1 \mathrm{E}\left\{\mathbf{M} | \boldsymbol{\alpha}'\mathbf{M} \leq Q_{\boldsymbol{\alpha}'\mathbf{M}}(p)\right\} p\varphi'(p)\, dp, \tag{5.235}$$

where $\mathbf{M}$ is the random vector (5.222) that represents the market. The investor will focus on the entries of the vector (5.235) that display a large absolute value.

For example, in the case of normal markets, from (5.229) we obtain directly the first-order derivatives:

$$\frac{\partial \operatorname{Spc}_{\varphi}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} = (\boldsymbol{\mu} - \mathbf{P}_T) + \frac{\boldsymbol{\Sigma}\boldsymbol{\alpha}}{\sqrt{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}} \mathcal{I}[\varphi z]. \tag{5.236}$$

In this expression z is the quantile of the standard normal distribution (5.228).

In particular in the case of the expected shortfall the spectrum is (5.218). From (B.50) we obtain:

$$\varphi'_{\mathrm{ES}_c}(p) = -\frac{\delta^{(1-c)}(p)}{1-c}, \tag{5.237}$$

where δ is the Dirac delta (B.17). Therefore (5.235) becomes the following expression:

$$\frac{\partial \operatorname{ES}_c}{\partial \boldsymbol{\alpha}} = \mathrm{E}\left\{\mathbf{M} | \boldsymbol{\alpha}'\mathbf{M} \leq \mathrm{Q}_c(\boldsymbol{\alpha})\right\}, \tag{5.238}$$

see Tasche (1999), Hallerbach (2003), Gourieroux, Laurent, and Scaillet (2000).

We remark that since spectral indices of satisfaction are positive homogenous, they satisfy the Euler decomposition (5.67). From (5.235) this reads:

$$\operatorname{Spc}_{\varphi}(\boldsymbol{\alpha}) = \sum_{n=1}^{N} \alpha_n \left[\int_0^1 \mathrm{E}\left\{M_n | \Psi_{\boldsymbol{\alpha}} \leq Q_{\Psi_\alpha}(p)\right\} [-p\varphi'(p)]\, dp\right]. \tag{5.239}$$

The contribution to satisfaction from each security in turn factors into the product of the amount of that security times the marginal contribution to satisfaction of that security. The marginal contribution to satisfaction (5.235) is insensitive to a rescaling of the portfolio, although it depends on the allocation.

The study of the second-order cross-derivatives provides insight on the local convexity/concavity of the certainty-equivalent. In Appendix www.5.5 we prove the following result:

$$\frac{\partial^2 \operatorname{Spc}_{\varphi}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\alpha}'} = \int_0^1 \operatorname{Cov}\left\{\mathbf{M} | \Psi_{\boldsymbol{\alpha}} = Q_{\Psi_\alpha}(p)\right\} f_{\Psi_\alpha}(Q_{\Psi_\alpha}(p))\, \varphi'(p)\, dp, \tag{5.240}$$

where f_{Ψ_α} is the marginal probability density function of the investor's objective $\Psi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\mathbf{M}$.

Since any covariance matrix and any probability density function are positive, whereas the derivative of the spectrum from (5.217) is negative, the second-order cross-derivatives define a negative definite matrix. Therefore the spectral indices of satisfaction are concave, see Figure 5.13, which refers to the expected shortfall.

For example, consider the case of normal markets. By direct derivation of (5.236) we obtain:

$$\frac{\partial^2 \operatorname{Spc}_{\varphi}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}' \partial \boldsymbol{\alpha}} = \boldsymbol{\Sigma} \left(\mathbf{I} - \frac{\boldsymbol{\alpha}\boldsymbol{\alpha}'\boldsymbol{\Sigma}}{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}} \right) \frac{\mathcal{I}[\varphi z]}{\sqrt{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}}. \tag{5.241}$$

In this expression z is the quantile of the standard normal distribution (5.228). Notice that the integral $\mathcal{I}[\varphi z]$ is negative, since the spectrum φ weighs the negative values of the quantile z more than the positive values. Therefore, by the same argument used in (5.192), the second derivative (5.241) is always negative definite and thus the spectral index of satisfaction is concave.

In particular, in the case of the expected shortfall, substituting (5.237) in (5.240), we obtain:

$$\frac{\partial^2 \operatorname{ES}_c(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\alpha}'} = -\frac{f_{\Psi_{\boldsymbol{\alpha}}}(\mathrm{Q}_c(\boldsymbol{\alpha}))}{1-c} \operatorname{Cov}\{\mathbf{M} | \Psi_{\boldsymbol{\alpha}} = \mathrm{Q}_c(\boldsymbol{\alpha})\}, \tag{5.242}$$

see Rau-Bredow (2002).