6

Optimizing allocations

In this chapter we determine the optimal allocation for a generic investor in a generic market of securities.

In Section 6.1 we introduce allocation optimization by means of a fully worked-out, non-trivial leading example. First of all, one must collect the necessary inputs. The first input is the investor's profile, namely his current allocation, his market, his investment horizon, his objectives and the respective indices of satisfaction that evaluate them. The second input is information on the investor's market, namely the joint distribution of the prices at the investment horizon of the pool of securities in the investor's market, as well as information on the implementation costs associated with trading those securities. By suitably processing the above inputs we can in principle derive the most suitable allocation for a given investor.

Except in our leading example, or in trivial cases of no practical interest, the constrained optimization that yields the optimal allocation cannot be solved analytically. In order to understand which optimization problems are numerically tractable, in Section 6.2 we present an overview of results on convex optimization, with particular focus on cone programming. First-order, second-order, and semidefinite cone programming encompass a broad class of constrained optimization problems that appear in the context of asset allocation.

In Section 6.3 we discuss a two-step approach that approximates the solution to the formal general allocation optimization by means of a tractable, quadratic problem. The first step in this approach is the mean-variance optimization pioneered by Markowitz, which selects a one-parameter family of efficient allocations among all the possible combinations of assets; the second step is a simple one-dimensional search for the best among the efficient allocations, which can be performed numerically. We introduce the mean-variance framework by means of geometrical arguments and discuss how to compute the necessary inputs that feed the mean-variance optimization. We also present the mean-variance problem in the less general, yet more common, formulation in terms of returns.

The mean-variance optimization admits analytical solutions in a broad class of cases, namely when the investment constraints are affine. In Section 6.4 we discuss these solutions, which provide insight into the properties of optimal portfolios in more general contexts. We discuss the two-fund separation theorem and study the effect of the market on the optimal allocations. Among other results, we prove wrong the common belief that a market with low correlations provides better investment opportunities than a highly correlated market.

The many advantages of the mean-variance approach at times obscure the few problems behind it. In Section 6.5 we analyze some pitfalls of this approach. We discuss the approximate nature of the mean-variance framework. We point out the nonsensical outcomes that can result from the common practice of considering the mean-variance formulation as an index of satisfaction. We highlight the conditions under which the mean-variance optimization represents a quadratic programming problem. We discuss the difference between the original mean-variance problem, which maximizes the expected value for a given level of variance, and the dual problem, which minimizes the variance for a given level of expected value. Finally, we discuss the drawbacks of presenting the mean-variance framework in terms of returns instead of prices. Indeed, returns can be used only under restrictive assumptions on the investor's preferences and constraints. Furthermore, even under those hypotheses, expressing the mean-variance problem in terms of returns leads to misinterpretations that dramatically affect the pursuit of the optimal allocation, for instance when the investment horizon is shifted in the future.

In Section 6.6 we show an application of the analytical solutions of the mean-variance problem: allocation against a benchmark. As it turns out, a benchmark is not only the explicit target of some fund managers but also the implicit target of the general investor. Indeed, we show how even so-called "total-return" investment strategies can be considered and analyzed as a special case of benchmark-driven allocation problems.

In Section 6.7 we conclude with a case study: allocation in the stock market. Unlike the leading example in Section 6.1, this case cannot be solved analytically. Therefore, we tackle it by means of the mean-variance approach. After revisiting the complete check-list of all the steps necessary to obtain the inputs of the mean-variance problem, we determine numerically the efficient allocations and then compute the optimal allocation by means of Monte Carlo simulations.

6.1 The general approach

Consider a market of N securities. We denote as $\mathbf{P}_t$ the prices at the generic time t of the N securities in the investor's market. At the time T when the investment is made, the investor can purchase α_n units of the generic n-th security. These units are specific to the security: for example in the case of equities the units are shares, in the case of futures the units are contracts, etc.

The N-dimensional vector $\boldsymbol{\alpha}$ represents the outcome of the allocation decision, which can be seen as a "black box" that processes two types of inputs: the information on the investor's profile $\mathcal{P}$ and the information i_T on the market available at the time the investment decision is made.

6.1.1 Collecting information on the investor

As far as the investor's profile $\mathcal{P}$ is concerned, information consists of knowledge of the investor's current situation and of his outlook.

The investor's current situation is summarized in his pre-existing, possibly null, portfolio $\boldsymbol{\alpha}^{(0)}$, which corresponds to his wealth, or endowment, at the time the investment decision is made:

$$w_T \equiv \mathbf{p}_T' \boldsymbol{\alpha}^{(0)}. \tag{6.1}$$

The lower-case notation for the prices at the investment date highlights the fact that these are deterministic quantities.

The investor's outlook includes first of all his choice of a market and of an investment horizon.

For example, for a private investor the market $\mathbf{P}_t$ could be a set of mutual funds and the investment horizon τ could be three years from the time the allocation decision is made.

Second, it is important to understand the investor's main objective $\Psi_{\boldsymbol{\alpha}}$, which depends on the allocation $\boldsymbol{\alpha}$. This could be final wealth as in (5.3), or relative wealth, as in (5.4), or net profits, as in (5.8), or possibly other specifications. Nevertheless, in any specification, the objective is a linear function of the allocation and of the market vector :

$$\Psi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}' \mathbf{M}, \tag{6.2}$$

see (5.10). The market vector $\mathbf{M}$ is a simple invertible affine transformation of the market prices at the investment horizon:

$$\mathbf{M} \equiv \mathbf{a} + \mathbf{B} \mathbf{P}_{T+\tau}, \tag{6.3}$$

where $\mathbf{a}$ is a suitable conformable vector and $\mathbf{B}$ is a suitable conformable invertible matrix, see (5.11).

For example, assume that the investor's main objective is final wealth. Then from (5.12) we obtain $\mathbf{M} \equiv \mathbf{P}_{T+\tau}$ and therefore:

$$\Psi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}' \mathbf{P}_{T+\tau}. \tag{6.4}$$

Third, we need to model the investor's attitude towards risk. This step is necessary because the markets, and thus the objective, are not deterministic. The investor's preferences are reflected in his index of satisfaction $\mathcal{S}$, see (5.48). The index of satisfaction depends on the allocation through the distribution of the investor's objective.

For instance the index of satisfaction could be the certainty-equivalent:

$$\mathcal{S}(\boldsymbol{\alpha}) \equiv \mathrm{CE}(\boldsymbol{\alpha}) \equiv u^{-1}\left(\mathrm{E}\left\{u\left(\Psi_{\boldsymbol{\alpha}}\right)\right\}\right). \tag{6.5}$$

Assume that the utility function belongs to the exponential class:

$$u(\psi) \equiv -e^{-\frac{1}{\zeta}\psi}, \tag{6.6}$$

where the risk propensity coefficient is a positive number comprised in a suitable interval:

$$\zeta \in \left[\underline{\zeta}, \overline{\zeta}\right]. \tag{6.7}$$

Under these specifications, from (5.94) we obtain the expression of the index of satisfaction:

$$\mathrm{CE}(\boldsymbol{\alpha}) = -\zeta \ln\left(\phi_{\Psi_{\boldsymbol{\alpha}}}\left(\frac{i}{\zeta}\right)\right), \tag{6.8}$$

where $\phi_{\Psi_{\boldsymbol{\alpha}}}$ is the characteristic function of the investor's objective.

In general an investor has multiple objectives. In other words, in addition to the main objective Ψ there exists a set of secondary objectives $\widetilde{\Psi}$ that the investor cares about. As for the main objective (6.2), any secondary objective is a linear function of the allocation and of its respective market vector:

$$\widetilde{\Psi}_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\widetilde{\mathbf{M}}. \tag{6.9}$$

As in (6.3), the market vector of a secondary objective is an affine transformation of the prices at the investment horizon:

$$\widetilde{\mathbf{M}} \equiv \widetilde{\mathbf{a}} + \widetilde{\mathbf{B}}\mathbf{P}_{T+\tau}. \tag{6.10}$$

A secondary objective is evaluated according to its specific index of satisfaction $\widetilde{\mathcal{S}}$, see (5.48).

In our example we assume that in addition to his level of final wealth (6.4), the investor is concerned about his net profits since the investment date. In this case from (5.14) the market vector reads $\widetilde{\mathbf{M}} \equiv \mathbf{P}_{T+\tau} - \mathbf{p}_T$ and the auxiliary objective, namely the net profits, reads:

$$\widetilde{\Psi}_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\left(\mathbf{P}_{T+\tau} - \mathbf{p}_T\right). \tag{6.11}$$

Furthermore, we assume that the investor evaluates his net profits in terms of their value at risk. In other words, other things equal the investor is happier if the value at risk of his investment is smaller. Therefore from (5.158)-(5.159) the index of satisfaction relative to the investor's net profits reads:

$$\widetilde{\mathcal{S}}(\boldsymbol{\alpha}) \equiv -\operatorname{Var}_c(\boldsymbol{\alpha}) \equiv Q_{\widetilde{\Psi}_\alpha}(1-c), \tag{6.12}$$

where Q denotes the quantile of the secondary objective (6.11) and c the VaR confidence level.

The investor's current portfolio, market, investment horizon, main objective and respective index of satisfaction, as well as his secondary objectives and respective indices of satisfaction, complete the check list of the information $\mathcal{P}$ regarding the investor's profile.

6.1.2 Collecting information on the market

As far as the market is concerned, it is important to collect information about the current prices of the securities $\mathbf{p}_T$ and their future values at the investment horizon $\mathbf{P}_{T+\tau}$.

The current prices $\mathbf{p}_T$ are deterministic variables that are publicly available at time T.

The future prices $\mathbf{P}_{T+\tau}$ are a random variable: therefore information on the prices at the investment horizon corresponds to information on their distribution. We recall that the distribution of the prices is obtained as follows: first we detect from time series analysis the invariants $\mathbf{X}_{t,\widetilde{\tau}}$ behind the market prices $\mathbf{P}_t$ relative to a suitable estimation horizon $\widetilde{\tau}$, see Section 3.1; then we estimate the distribution of the invariants $\mathbf{X}_{t,\widetilde{\tau}}$ see Chapter 4; next, we project these invariants $\mathbf{X}_{t,\widetilde{\tau}}$ to the investment horizon, obtaining the distribution of $\mathbf{X}_{T+\tau,\tau}$, see Section 3.2; finally we map the distribution of the invariants $\mathbf{X}_{T+\tau,\tau}$ into the distribution of the prices at the investment horizon of the securities $\mathbf{P}_{T+\tau}$, see Section 3.3.

For example, we assume that the distribution of the prices at the investment horizon is estimated to be normal:

$$\mathbf{P}_{T+\tau} \sim \mathrm{N}(\boldsymbol{\xi}, \boldsymbol{\Phi}), \tag{6.13}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\Phi}$ are suitable values respectively of the expected value and covariance matrix of the market prices.

Furthermore, the process of switching from a generic allocation $\widetilde{\boldsymbol{\alpha}}$ to another generic allocation $\boldsymbol{\alpha}$ is costly. We denote as $\mathcal{T}(\widetilde{\boldsymbol{\alpha}}, \boldsymbol{\alpha})$ the *transaction costs* associated with this process. Transaction costs take different forms in different markets: for instance, traders face bid-ask spreads and commissions, private investors face subscription fees, etc.

For example, the investor might be charged a commission that is proportional to the number of securities transacted:

$$\mathcal{T}(\widetilde{\boldsymbol{\alpha}}, \boldsymbol{\alpha}) \equiv \mathbf{k}' |\widetilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}|, \tag{6.14}$$

where $\mathbf{k}$ is a given constant vector. In the sequel of our example we consider the simplified case of null transaction costs, i.e. $\mathbf{k} \equiv \mathbf{0}$.

The current prices of the securities in the market, the distribution of the market at the investment horizon and the details on the transaction costs complete the check list of the information i_T regarding the market.

6.1.3 Computing the optimal allocation

A generic allocation decision processes the information on the market and on the investor and outputs the amounts to invest in each security in the given market:

$$\boldsymbol{\alpha}[\cdot]: \ [i_T, \mathcal{P}] \mapsto \mathbb{R}^N. \tag{6.15}$$

We stress that this is the definition of a generic allocation decision, not necessarily optimal.

For example, a possible decision allocates an equal amount of the initial wealth w_T in each security in the market. This is the *equally-weighted portfolio*:

$$\boldsymbol{\alpha}[i_T, \mathcal{P}] \equiv \frac{w_T}{N} \operatorname{diag}(\mathbf{p}_T)^{-1} \mathbf{1}_N, \tag{6.16}$$

where $\mathbf{1}$ denotes a vector of ones. Notice that this decision uses very little information on the market, i.e. only the current prices of the securities, and very little information on the investor, i.e. only his initial budget.

In order to be optimal, an allocation decision ensues from carefully processing all the available information about both the investor and the market.

First of all, given the distribution of the prices at the investment horizon $\mathbf{P}_{T+\tau}$, it is possible in principle to compute explicitly the distribution of the investor's generic objective Ψ as a function of the allocation.

Indeed, the market vector $\mathbf{M}$ that correspond to the investor's objective Ψ as in (6.2) or (6.9) is a simple invertible affine transformation of the prices at the investment horizon, see (6.3) and (6.10). Therefore the distribution of the market vector $\mathbf{M}$ is easily obtained from the distribution of $\mathbf{P}_{T+\tau}$, see (5.15). Furthermore, since the objective $\Psi \equiv \boldsymbol{\alpha}'\mathbf{M}$ is a linear combination of the allocation and the market vector, we can in principle determine its distribution from the distribution of $\mathbf{M}$.

In our example, from (6.13) the market vector relative to the main objective (6.4) is normally distributed:

$$\mathbf{M} \equiv \mathbf{P}_{T+\tau} \sim \mathrm{N}\left(\boldsymbol{\xi}, \boldsymbol{\Phi}\right). \tag{6.17}$$

Therefore the investor's main objective is normally distributed with the following parameters:

$$\Psi_{\boldsymbol{\alpha}} \sim \mathrm{N}\left(\boldsymbol{\xi}'\boldsymbol{\alpha}, \boldsymbol{\alpha}'\boldsymbol{\Phi}\boldsymbol{\alpha}\right). \tag{6.18}$$

Similarly, from (6.13) the market vector relative to the secondary objective (6.11) is normally distributed:

$$\widetilde{\mathbf{M}} \equiv \mathbf{P}_{T+\tau} - \mathbf{p}_T \sim \mathrm{N}\left(\boldsymbol{\xi} - \mathbf{p}_T, \boldsymbol{\Phi}\right). \tag{6.19}$$

Therefore the investor's secondary objective is normally distributed with the following parameters:

$$\widetilde{\Psi}_{\boldsymbol{\alpha}} \sim \mathrm{N}\left(\left(\boldsymbol{\xi} - \mathbf{p}_T\right)' \boldsymbol{\alpha}, \boldsymbol{\alpha}'\boldsymbol{\Phi}\boldsymbol{\alpha}\right). \tag{6.20}$$

From the distribution of the investor's primary and secondary objectives we can compute the respective indices of satisfaction.

In our example, substituting the characteristic function (1.69) of the first objective (6.18) in (6.8) we obtain the main index of satisfaction, namely the certainty-equivalent of final wealth:

$$\mathrm{CE}\left(\boldsymbol{\alpha}\right) = \boldsymbol{\xi}'\boldsymbol{\alpha} - \frac{1}{2\zeta}\boldsymbol{\alpha}'\boldsymbol{\Phi}\boldsymbol{\alpha}. \tag{6.21}$$

Similarly, substituting the quantile (1.70) of the secondary objective (6.20) in (6.12) we obtain the (opposite of the) secondary index of satisfaction, namely the value at risk:

$$\mathrm{Var}_c\left(\boldsymbol{\alpha}\right) = \left(\mathbf{p}_T - \boldsymbol{\xi}\right)' \boldsymbol{\alpha} + \sqrt{2\boldsymbol{\alpha}'\boldsymbol{\Phi}\boldsymbol{\alpha}}\,\mathrm{erf}^{-1}\left(2c - 1\right), \tag{6.22}$$

where erf^{-1} is the inverse of the error function (B.75).

Finally, the investor is bound by a set of investment constraints $\mathcal{C}$ that limit his feasible allocations.

One constraint that appears in different forms is the *budget constraint*, which states that the value of the initial investment cannot exceed a given budget b net of transaction costs:

$$\mathcal{C}_1: \ \mathbf{p}_T'\boldsymbol{\alpha} + \mathcal{T}\left(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}\right) - b \leq 0, \tag{6.23}$$

where $\boldsymbol{\alpha}^{(0)}$ is the initial portfolio (6.1).

Notice that we defined the constraint in terms of an inequality. In most applications in the final optimal allocation this constraint turns out to be binding, i.e. it is satisfied as an equality.

In our example the transaction costs (6.14) are null, and we assume that the budget is the initial wealth. Therefore the budget constraint reads:

$$\mathcal{C}_1: \ \mathbf{p}_T' \boldsymbol{\alpha} = w_T \equiv \mathbf{p}_T' \boldsymbol{\alpha}^{(0)}. \tag{6.24}$$

By means of additional constraints we can include the investor's multiple objectives in the allocation problem. Indeed, the multiple objectives are accounted for by imposing that the respective index of satisfaction $\widetilde{\mathcal{S}}$ exceed a minimum acceptable threshold $\widetilde{s}$:

$$\mathcal{C}_2: \ \widetilde{s} - \widetilde{\mathcal{S}}(\boldsymbol{\alpha}) \leq 0. \tag{6.25}$$

In our example the additional objective of the investor are his net profits (6.11), which the investor monitors by means of the value at risk (6.12). An allocation is acceptable for the investor only if the respective VaR does not exceed a given *budget at risk*, i.e. a fraction γ of the initial endowment:

$$\mathcal{C}_2: \ \mathrm{Var}_c(\boldsymbol{\alpha}) \leq \gamma w_T. \tag{6.26}$$

From (6.22) and (6.24) the VaR constraint reads explicitly:

$$\mathcal{C}_2: \ (1-\gamma) w_T - \boldsymbol{\xi}' \boldsymbol{\alpha} + \sqrt{2 \boldsymbol{\alpha}' \boldsymbol{\Phi} \boldsymbol{\alpha}} \, \mathrm{erf}^{-1}(2c-1) \leq 0. \tag{6.27}$$

We denote an allocation that satisfies the given set of constraints $\mathcal{C}$ as follows:

$$\boldsymbol{\alpha} \in \mathcal{C}. \tag{6.28}$$

The set of allocations that satisfy the constraints is called the *feasible set*.

To determine the feasible set in our example, we consider the plane of coordinates:

$$e \equiv \boldsymbol{\xi}' \boldsymbol{\alpha}, \quad d \equiv \sqrt{\boldsymbol{\alpha}' \boldsymbol{\Phi} \boldsymbol{\alpha}}. \tag{6.29}$$

As we show in Figure 6.1, in this plane the budget constraint (6.24) is satisfied by all the points in the region to the right of a hyperbola:

$$d^2 \geq \frac{A}{D} e^2 - \frac{2 w_T B}{D} e + \frac{w_T^2 C}{D}, \tag{6.30}$$

where

$$\begin{array}{ll} A \equiv \mathbf{p}_T' \boldsymbol{\Phi}^{-1} \mathbf{p}_T & B \equiv \mathbf{p}_T' \boldsymbol{\Phi}^{-1} \boldsymbol{\xi} \\ C \equiv \boldsymbol{\xi}' \boldsymbol{\Phi}^{-1} \boldsymbol{\xi} & D \equiv AC - B^2, \end{array} \tag{6.31}$$

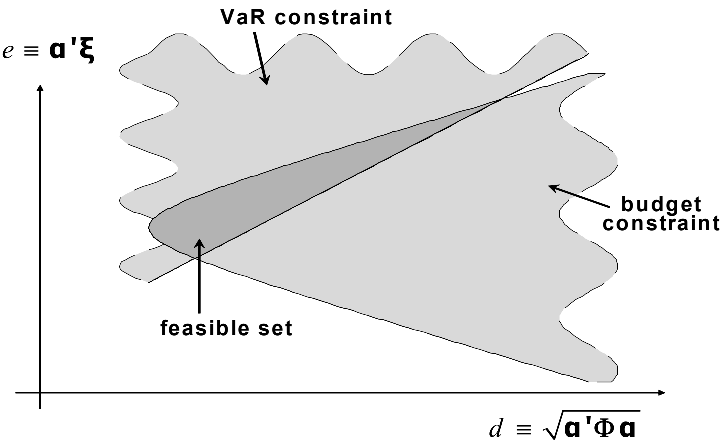

Fig. 6.1. Leading allocation example: constraints and feasible set

see Appendix www.6.1 for the proof.

On the other hand, the VaR constraint (6.27) is satisfied by all the points above a straight line:

$$e \geq (1-\gamma)\, w_T + \sqrt{2}\, \mathrm{erf}^{-1}(2c-1)\, d. \tag{6.32}$$

This follows immediately from (6.27).

The investor evaluates the potential advantages of an allocation $\boldsymbol{\alpha}$ based on his primary index of satisfaction $\mathcal{S}$, provided that the allocation is feasible. Therefore, the optimal allocation is the solution to the following maximization problem:

$$\boldsymbol{\alpha}^* \equiv \underset{\boldsymbol{\alpha} \in \mathcal{C}}{\operatorname{argmax}} \left\{ \mathcal{S}(\boldsymbol{\alpha}) \right\}. \tag{6.33}$$

By construction, the optimal allocation is of the form (6.15), i.e. it is a "black box" that processes both the current information on the market i_T and the investor's profile $\mathcal{P}$, and outputs a vector of amounts of each security in the market.

In our example, from (6.5), (6.24) and (6.26) the investor solves:

$$\boldsymbol{\alpha}^* \equiv \underset{\substack{\mathbf{p}_T' \boldsymbol{\alpha} = w_T \\ \mathrm{Var}_c(\boldsymbol{\alpha}) \leq \gamma w_T}}{\operatorname{argmax}} \left\{ \mathrm{CE}(\boldsymbol{\alpha}) \right\}. \tag{6.34}$$

To compute the optimal solution we consider the plane of the following coordinates (see Figure 6.2):

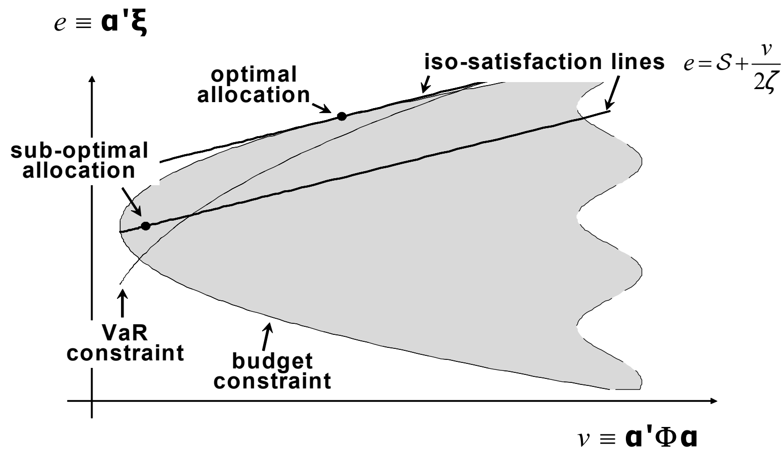

Fig. 6.2. Leading allocation example: optimal allocation

$$e \equiv \boldsymbol{\xi}'\boldsymbol{\alpha}, \quad v \equiv \boldsymbol{\alpha}'\boldsymbol{\Phi}\boldsymbol{\alpha}. \tag{6.35}$$

In these coordinates the allocations that satisfy the budget constraint plot to the right of a parabola:

$$v \geq \frac{A}{D}e^2 - \frac{2w_T B}{D}e + \frac{w_T^2 C}{D}. \tag{6.36}$$

This follows immediately from (6.30). Similarly the allocations that satisfy the VaR constraint plot to the left of a parabola:

$$v \leq \frac{1}{2\left(\operatorname{erf}^{-1}(2c-1)\right)^2}\left(e - (1-\gamma)\,w_T\right)^2. \tag{6.37}$$

This follows immediately from (6.32).

From (6.21) the iso-satisfaction contours, i.e. the allocations that give rise to the same level of satisfaction $\mathcal{S}$, plot along the following straight line:

$$e = \mathcal{S} + \frac{v}{2\zeta}. \tag{6.38}$$

The slope of this straight line is determined by the risk propensity coefficient ζ: the higher this coefficient, the flatter the line. The level of this straight line is determined by the amount of satisfaction: the larger $\mathcal{S}$, the higher the plot of the line (6.38) in the plane.

To solve for the optimal allocation we need to determine the highest iso-satisfaction line that has an intersection with the region to the right of the budget-constraint parabola (6.36), namely the tangent line to the upper branch of that parabola, see Figure 6.2. As long as the coefficient ζ is comprised between suitable limits as in (6.7), the optimal allocation automatically satisfies the VaR constraint.

We prove in Appendix www.6.1 that the optimal allocation reads:

$$\boldsymbol{\alpha}^* \equiv \zeta \boldsymbol{\Phi}^{-1} \boldsymbol{\xi} + \frac{w_T - \zeta \mathbf{p}_T' \boldsymbol{\Phi}^{-1} \boldsymbol{\xi}}{\mathbf{p}_T' \boldsymbol{\Phi}^{-1} \mathbf{p}_T} \boldsymbol{\Phi}^{-1} \mathbf{p}_T. \tag{6.39}$$

The optimal allocation is of the form (6.15). Indeed, the information on the market is summarized in the current prices and in expected values and covariances of the future prices at the investment horizon:

$$i_T \equiv (\mathbf{p}_T, \boldsymbol{\xi}, \boldsymbol{\Phi}); \tag{6.40}$$

and the investor's profile is summarized in his risk propensity and initial budget:

$$\mathcal{P} \equiv (\zeta, w_T). \tag{6.41}$$

The allocation (6.39) gives rise to the maximum level of satisfaction, given the investment constraints. We prove in Appendix www.6.1 that this level reads:

$$\begin{aligned} \mathrm{CE}(\boldsymbol{\alpha}^*) = & \frac{\zeta}{2} \boldsymbol{\xi}' \boldsymbol{\Phi}^{-1} \boldsymbol{\xi} + \frac{1}{2} \left(\frac{w_T - \zeta \mathbf{p}_T' \boldsymbol{\Phi}^{-1} \boldsymbol{\xi}}{\mathbf{p}_T' \boldsymbol{\Phi}^{-1} \mathbf{P}_T} \right) \boldsymbol{\xi}' \boldsymbol{\Phi}^{-1} \mathbf{p}_T \\ & - \frac{1}{2\zeta} \frac{\left(w_T - \zeta \mathbf{p}_T' \boldsymbol{\Phi}^{-1} \boldsymbol{\xi} \right)^2}{\mathbf{p}_T' \boldsymbol{\Phi}^{-1} \mathbf{p}_T}. \end{aligned} \tag{6.42}$$

6.2 Constrained optimization

From (6.33) we see that determining the best allocation for a given investor boils down to solving a constrained optimization problem. In this section we present a quick review of results on constrained optimization. The reader is referred to Lobo, Vandenberghe, Boyd, and Lebret (1998), Ben-Tal and Nemirovski (2001), Boyd and Vandenberghe (2004) and references therein for more on this subject.

Consider a generic constrained optimization problem:

$$\mathbf{z}^* \equiv \underset{\mathbf{z} \in \mathcal{C}}{\operatorname{argmin}} \, \mathcal{Q}(\mathbf{z}), \tag{6.43}$$

where $\mathcal{Q}$ is the objective function, and $\mathcal{C}$ is a set of constraints. Here, following the standards of the literature, we present optimization as a minimization problem. To consider maximization problems, it suffices to change the sign of the objective, turning the respective problem into a minimization.

In general it is not possible to solve (6.43) analytically. Nonetheless, even within the realm of numerical optimization, not all problems can be solved.

A broad class of constrained optimization problems that admit numerical solutions is represented by *convex programming* problems: in this framework the objective $\mathcal{Q}$ is a convex function and the feasible set determined by the constraints is the intersection of a hyperplane and a convex set. More precisely, convex programming is an optimization problem of the form:

$$\mathbf{z}^{*} \equiv \underset{\substack{\mathbf{z}\in\mathcal{L} \\ \mathbf{z}\in\mathcal{V}}}{\operatorname{argmin}}\, \mathcal{Q}\left(\mathbf{z}\right), \tag{6.44}$$

where $\mathcal{Q}$ is a convex function, i.e. it satisfies (5.82); $\mathcal{L}$ is a hyperplane determined by a conformable matrix $\mathbf{A}$ and a conformable vector $\mathbf{a}$:

$$\mathcal{L} \equiv \{\mathbf{z} \text{ such that } \mathbf{Az} = \mathbf{a}\}\,; \tag{6.45}$$

and $\mathcal{V}$ is a convex set, determined implicitly by a set of inequalities on convex functions:

$$\mathcal{V} \equiv \{\mathbf{z} \text{ such that } \mathbf{F}\left(\mathbf{z}\right) \leq \mathbf{0},\, \mathbf{F} \text{ convex}\}\,. \tag{6.46}$$

Although numerical solutions can be found for convex programming, these are usually computationally too expensive for the amount of variables involved in an asset allocation problem.

Nonetheless, a subclass of convex programming, called *cone programming* (CP), admits efficient numerical solutions, which are variations of *interior point algorithms*, see Nesterov and Nemirovski (1995). In conic programming the objective is linear, and the feasible set determined by the constraints is the intersection of a hyperplane and a cone. More precisely, convex programming is an optimization problem of the form:

$$\mathbf{z}^{*} \equiv \underset{\substack{\mathbf{z}\in\mathcal{L} \\ \mathbf{Bz}-\mathbf{b}\in\mathcal{K}}}{\operatorname{argmin}}\, \mathbf{c}'\mathbf{z}, \tag{6.47}$$

where $\mathbf{c}$ and $\mathbf{b}$ are conformable vectors; $\mathbf{B}$ is a conformable matrix; $\mathcal{L}$ is a hyperplane as in (6.45); and $\mathcal{K}$ is a *cone*, i.e. a set with the following properties, see Figure 6.3:

i. a cone is closed under positive multiplication, i.e. it extends to infinity in radial directions from the origin:

$$\mathbf{y} \in \mathcal{K}, \lambda \geq 0 \Rightarrow \lambda\mathbf{y} \in \mathcal{K}; \tag{6.48}$$

ii. a cone is closed under addition, i.e. it includes all its interior points:

$$\mathbf{y}, \widetilde{\mathbf{y}} \in \mathcal{K} \Rightarrow \mathbf{y} + \widetilde{\mathbf{y}} \in \mathcal{K}; \tag{6.49}$$

iii. a cone is "pointed", i.e. it lies on only one side of the origin. More formally, for any $\mathbf{y} \neq \mathbf{0}$ the following holds:

$$\mathbf{y} \in \mathcal{K} \Rightarrow -\mathbf{y} \notin \mathcal{K}. \tag{6.50}$$

Depending on what type of cone $\mathcal{K}$ defines the constraints in the conic programming (6.47), we obtain as special cases all the problems that currently can be solved. In particular, we distinguish three types of cones, and the respective notable classes of conic programming.

6.2.1 Positive orthants: linear programming

Consider the positive orthant of dimension M, i.e. the subset of $\mathbb{R}^M$ spanned by the positive coordinates:

$$\mathbb{R}_+^M \equiv \left\{ \mathbf{y} \in \mathbb{R}^M \text{ such that } y_1 \geq 0, \ldots, y_M \geq 0 \right\}. \tag{6.51}$$

It is easy to check that the positive orthant $\mathbb{R}_+^M$ is a cone, i.e. it satisfies (6.48)-(6.50). The ensuing conic programming (6.47) problem reads:

$$\mathbf{z}^* \equiv \underset{\substack{\mathbf{Az}=\mathbf{a} \\ \mathbf{Bz}\geq\mathbf{b}}}{\operatorname{argmin}}\, \mathbf{c}'\mathbf{z}. \tag{6.52}$$

This problem is called *linear programming* (LP), see Dantzig (1998).

6.2.2 Ice-cream cones: second-order cone programming

Consider the *ice-cream cone*, or *Lorentz cone*, of dimension M:

$$\mathbb{K}^M \equiv \left\{ \mathbf{y} \in \mathbb{R}^M \text{ such that } \left\| (y_1, \ldots, y_{M-1})' \right\| \leq y_M \right\}, \tag{6.53}$$

where $\|\cdot\|$ is the standard norm $(A.6)$ in $\mathbb{R}^M$, see Figure 6.3.

It is easy to check that the ice-cream cone $\mathbb{K}^M$ is indeed a cone, i.e. it satisfies (6.48)-(6.50). Furthermore, the direct product of ice-cream cones is also cone:

$$\begin{aligned} \mathcal{K} &\equiv \mathbb{K}^{M_1} \times \cdots \times \mathbb{K}^{M_J} \\ &\equiv \left\{ \mathbf{y}_{(1)} \in \mathbb{K}^{M_1} \times \cdots \times \mathbf{y}_{(J)} \in \mathbb{K}^{M_J} \right\}. \end{aligned} \tag{6.54}$$

The ensuing conic programming problem (6.47) reads:

$$\begin{aligned} \mathbf{z}^* &\equiv \underset{\mathbf{z}}{\operatorname{argmin}}\, \mathbf{c}'\mathbf{z} \\ &\text{subject to } \begin{cases} \mathbf{Az} = \mathbf{a} \\ \left\| \mathbf{D}_{(1)}\mathbf{z} - \mathbf{q}_{(1)} \right\| \leq \mathbf{p}'_{(1)}\mathbf{z} - r_{(1)} \\ \vdots \\ \left\| \mathbf{D}_{(J)}\mathbf{z} - \mathbf{q}_{(J)} \right\| \leq \mathbf{p}'_{(J)}\mathbf{z} - r_{(J)}, \end{cases} \end{aligned} \tag{6.55}$$

where $\mathbf{D}_{(j)}$ are conformable matrices, $\mathbf{q}_{(j)}$ are conformable vectors and $r_{(j)}$ are scalars for $j = 1, \ldots, J$. This follows by defining the matrix $\mathbf{B}$ and the vector $\mathbf{b}$ in (6.47) as below:

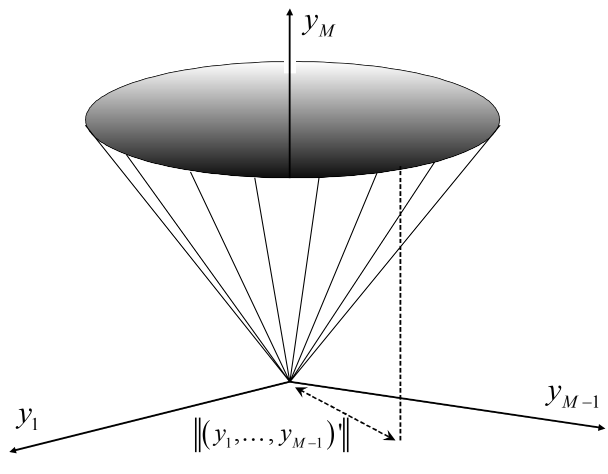

Fig. 6.3. Lorentz cone

$$\mathbf{B} \equiv \begin{pmatrix} \mathbf{D}_{(1)} \\ \mathbf{p}'_{(1)} \\ \vdots \\ \mathbf{D}_{(J)} \\ \mathbf{p}'_{(J)} \end{pmatrix}, \quad \mathbf{b} \equiv \begin{pmatrix} \mathbf{q}_{(1)} \\ r_{(1)} \\ \vdots \\ \mathbf{q}_{(J)} \\ r_{(J)} \end{pmatrix}. \tag{6.56}$$

The optimization problem (6.55) is called *second-order cone programming* (SOCP).

Second-order cone programming problems include *quadratically constrained quadratic programming* (QCQP) problems as a subclass. A generic QCQP problem reads:

$$\mathbf{z}^* \equiv \underset{\mathbf{z}}{\operatorname{argmin}} \left\{ \mathbf{z}'\mathbf{S}_{(0)}\mathbf{z} + 2\mathbf{u}'_{(0)}\mathbf{z} + v_{(0)} \right\} \tag{6.57}$$

$$\text{subject to } \begin{cases} \mathbf{Az} = \mathbf{a} \\ \mathbf{z}'\mathbf{S}_{(1)}\mathbf{z} + 2\mathbf{u}'_{(1)}\mathbf{z} + v_{(1)} \leq 0 \\ \vdots \\ \mathbf{z}'\mathbf{S}_{(J)}\mathbf{z} + 2\mathbf{u}'_{(J)}\mathbf{z} + v_{(J)} \leq 0, \end{cases}$$

where $\mathbf{S}_{(j)}$ are symmetric and positive matrices, $\mathbf{u}_{(j)}$ are conformable vectors and $v_{(j)}$ are scalars for $j = 0, \ldots, J$. Consider the spectral decomposition $(A.66)$ of the matrices $\mathbf{S}_{(j)}$:

$$\mathbf{S}_{(j)} \equiv \mathbf{E}_{(j)}\boldsymbol{\Lambda}_{(j)}\mathbf{E}'_{(j)}, \tag{6.58}$$

where $\boldsymbol{\Lambda}$ is the diagonal matrix of the eigenvalues and $\mathbf{E}$ is the juxtaposition of the respective eigenvectors. As we show in Appendix www.6.2, by introducing an auxiliary variable t the QCQP problem (6.57) can be written equivalently as follows:

$$(\mathbf{z}^*, t^*) \equiv \underset{(\mathbf{z},t)}{\operatorname{argmin}}\, t \tag{6.59}$$
$$\text{s.t.} \begin{cases} \mathbf{Az} = \mathbf{a} \\ \left\| \boldsymbol{\Lambda}_{(0)}^{1/2} \mathbf{E}'_{(0)} \mathbf{z} + \boldsymbol{\Lambda}_{(0)}^{-1/2} \mathbf{E}'_{(0)} \mathbf{u}_{(0)} \right\| \leq t \\ \left\| \boldsymbol{\Lambda}_{(1)}^{1/2} \mathbf{E}'_{(1)} \mathbf{z} + \boldsymbol{\Lambda}_{(1)}^{-1/2} \mathbf{E}'_{(1)} \mathbf{u}_{(1)} \right\| \leq \sqrt{\mathbf{u}_{(1)} \mathbf{S}_{(1)}^{-1} \mathbf{u}_{(1)} - v_{(1)}} \\ \vdots \\ \left\| \boldsymbol{\Lambda}_{(J)}^{1/2} \mathbf{E}'_{(J)} \mathbf{z} + \boldsymbol{\Lambda}_{(J)}^{-1/2} \mathbf{E}'_{(J)} \mathbf{u}_{(J)} \right\| \leq \sqrt{\mathbf{u}_{(J)} \mathbf{S}_{(J)}^{-1} \mathbf{u}_{(J)} - v_{(J)}}. \end{cases}$$

This problem is in the SOCP form (6.55).

Quite obviously, the QCQP problem (6.57) also includes linearly constrained quadratic programming problems (QP) and linear programming problems (LP) as special cases.

6.2.3 Semidefinite cones: semidefinite programming

Consider the following set of $M \times M$ matrices:

$$\mathbb{S}_+^M \equiv \{\mathbf{S} \succeq \mathbf{0}\}, \tag{6.60}$$

where $\mathbf{S} \succeq \mathbf{0}$ denotes symmetric and positive. It is easy to check that this set is a cone, i.e. it satisfies (6.48)-(6.50). The cone $\mathbb{S}_+^M$ is called the *semidefinite cone*. The ensuing conic programming problem (6.47) reads:

$$\mathbf{z}^* \equiv \underset{\mathbf{z}}{\operatorname{argmin}}\, \mathbf{c}'\mathbf{z} \tag{6.61}$$
$$\text{s.t.} \begin{cases} \mathbf{Az} = \mathbf{a} \\ \mathbf{B}_{(1)} z_1 + \cdots + \mathbf{B}_{(N)} z_N - \mathbf{B}_{(0)} \succeq \mathbf{0}, \end{cases}$$

where $\mathbf{B}_{(j)}$ are symmetric, but not necessarily positive, matrices, for $j = 0, \ldots, N$.

The optimization problem (6.61) is called *semidefinite programming* (SDP). It is possible to show that SDP includes the SOCP problem (6.55) as a special case, see Lobo, Vandenberghe, Boyd, and Lebret (1998). Nonetheless, the computational cost to solve generic SDP problems is much higher than the cost to solve SOCP problems.

6.3 The mean-variance approach

Consider the general formalization (6.33) of an allocation optimization:

$$\boldsymbol{\alpha}^* \equiv \underset{\boldsymbol{\alpha}\in\mathcal{C}}{\operatorname{argmax}}\, \mathcal{S}(\boldsymbol{\alpha}). \tag{6.62}$$

In general it is not possible to determine the analytical solution of this problem. Indeed, the leading example detailed in Section 6.1 probably represents the only non-trivial combination of market, preferences and constraints that gives rise to a problem which can be solved analytically in all its steps. Therefore, we need to turn to numerical results.

Even within the domain of numerical solutions, if the primary index of satisfaction $\mathcal{S}$ targeted by the investor is not concave, or if the secondary indices of satisfaction $\widetilde{\mathcal{S}}$ that determine the constraints as in (6.25) are not concave, then the general allocation optimization problem (6.62) is not convex. Therefore, the allocation optimization problem is not as in (6.44) and thus it cannot be solved numerically. For instance, this happens when one among the primary or secondary indices of satisfaction is based on a quantile of the investor's objective (value at risk) or it based on its expected utility (certainty-equivalent), see (5.153) and (5.191). Numerical solutions cannot be computed in general for non-convex problems.

Furthermore, even if the allocation optimization problem (6.62) is convex, the computational cost of obtaining a solution is in general prohibitive: only the special class of conic programming problems can be computed efficiently, see Section 6.2. Therefore it is important to cast the general allocation optimization in this class, possibly by means of approximations.

In this section we discuss a two-step approximation to the general allocation optimization problem that is both intuitive and computationally tractable, namely the mean-variance approach. The mean-variance two-step approach is by far the most popular approach to asset allocation: it has become the guideline in all practical applications and the benchmark in all academic studies on the subject.

6.3.1 The geometry of allocation optimization

To better understand the generality and the limitations of the mean-variance approach we analyze the optimization problem (6.62) from a geometrical point of view.

In any of the formulations considered in Chapter 5, the investor's index of satisfaction is law invariant, i.e. it is a functional of the distribution of the investor's objective, see (5.52). In turn, the distribution of the investor's objective is in general univocally determined by its moments, see Appendix www.1.6. Therefore the index of satisfaction can be re-written as a function defined on the infinite-dimensional space of the moments of the distribution of the objective:

$$\mathcal{S}(\boldsymbol{\alpha}) \equiv \mathcal{H}\left(\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\}, \mathrm{CM}_2\{\Psi_{\boldsymbol{\alpha}}\}, \mathrm{CM}_3\{\Psi_{\boldsymbol{\alpha}}\}, \ldots\right). \tag{6.63}$$

In this expression CM_k denotes as in (1.48) the central moment of order k of a univariate distribution:

$$\mathrm{CM}_k\left\{\Psi\right\} \equiv \mathrm{E}\left\{\left(\Psi - \mathrm{E}\left\{\Psi\right\}\right)^k\right\}. \tag{6.64}$$

We chose to represent (6.63) in terms of the central moments, but we could equivalently have chosen the raw moments (1.47). The explicit functional expression of $\mathcal{H}$ in (6.63) is determined by the specific index of satisfaction $\mathcal{S}$ adopted to model the investor's preferences. For instance, when the index of satisfaction is the certainty-equivalent, the functional expression follows from a Taylor expansion of the utility function, see (5.146). When the index of satisfaction is a quantile or a spectral index, this expression follows from the Cornish-Fisher expansion, see (5.180) and (5.232).

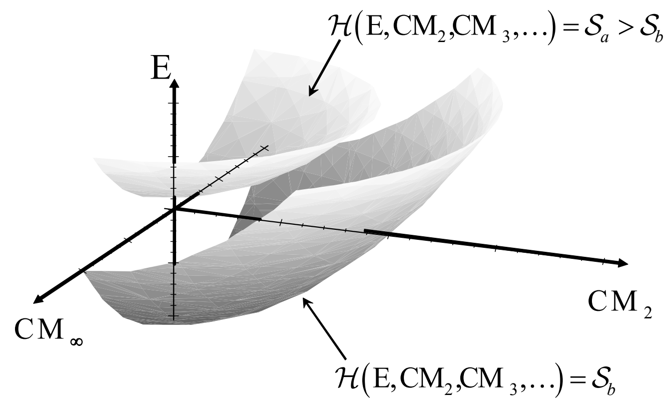

Fig. 6.4. Iso-satisfaction surfaces in the space of moments of the investor's objective

The iso-satisfaction surfaces in this space, i.e. the combinations of moments of the investor's objective that elicit an equal level of satisfaction, are defined by implicit equations as the following:

$$\mathcal{H}\left(\mathrm{E}, \mathrm{CM}_2, \mathrm{CM}_3, \ldots\right) = \mathcal{S}, \tag{6.65}$$

where $\mathcal{S}$ is a given constant level of satisfaction. Therefore, iso-satisfaction surfaces are "$(\infty - 1)$"-dimensional objects in the ∞-dimensional space of the moments of the investor's objective.

We represent this situation in Figure 6.4, where the ∞-dimensional space of moments is reduced to three dimensions and the $(\infty - 1)$-dimensional iso-satisfaction surfaces are represented by two-dimensional surfaces.

On the other hand, not all points in the space of the moments of the investor's objective correspond to an allocation. Indeed, as the allocation vector $\boldsymbol{\alpha}$ spans $\mathbb{R}^N$, the corresponding moments describe an N-dimensional surface $\mathcal{G}$ in the ∞-dimensional space of moments:

$$\mathcal{G}: \ \boldsymbol{\alpha} \mapsto \left(\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\}, \mathrm{CM}_2\left\{\Psi_{\boldsymbol{\alpha}}\right\}, \mathrm{CM}_3\left\{\Psi_{\boldsymbol{\alpha}}\right\}, \ldots\right). \tag{6.66}$$

In Figure 6.5 we sketch the case of $N \equiv 2$ securities: the shaded square represents $\mathbb{R}^N$ and the two-dimensional shape in the space of moments represents the combinations of moments that can be generated by an allocation.

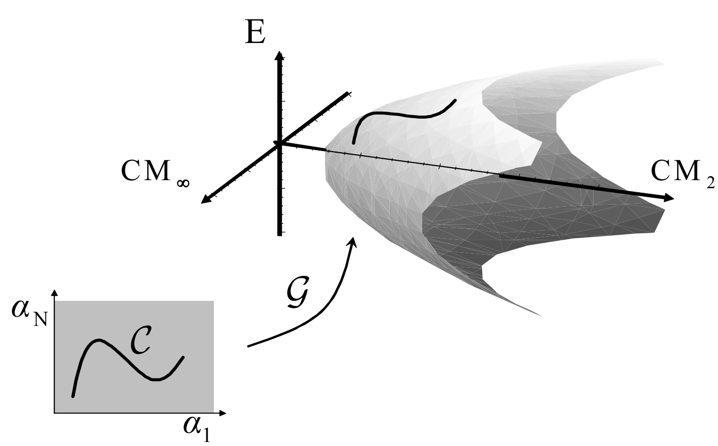

Fig. 6.5. Feasible allocations in the space of moments of the investor's objective

Finally, even within the surface (6.66), not all the allocations are viable, because the generic allocation $\boldsymbol{\alpha} \in \mathbb{R}^N$ has to satisfy a set of investment constraints, see (6.28). Therefore the feasible set becomes a subset of $\mathbb{R}^N$. This is reflected in the space of moments: the function $\mathcal{G}$ maps the feasible set into a lower-dimensional/lower-size portion of the N-dimensional surface described by (6.66).

We sketch this phenomenon in Figure 6.5 in our example of $N \equiv 2$ securities: the feasible set becomes a line, which is then mapped by $\mathcal{G}$ into the space of moments.

Solving the allocation optimization problem (6.62) corresponds to determining an iso-satisfaction surface that contains feasible points in the space

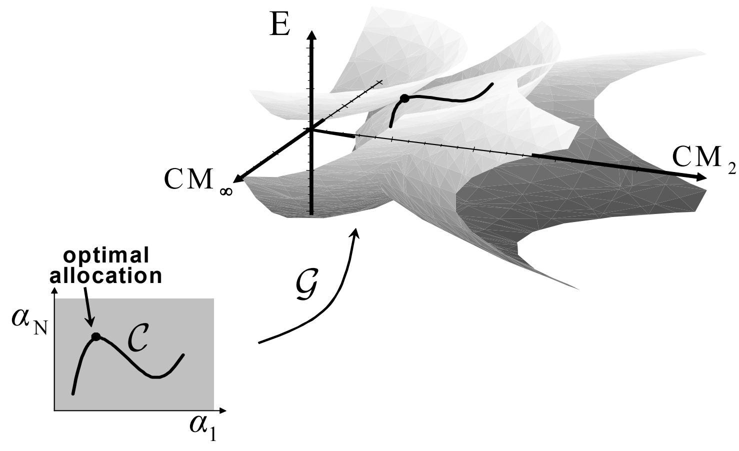

Fig. 6.6. Optimal allocation maximizes satsifaction

of moments and that corresponds to the highest possible level of satisfaction. Those feasible points corresponds to specific allocations that maximize satisfaction.

In Figure 6.6 the highest possible level of satisfaction compatible with the constraints in the space of moments corresponds to a specific allocation in the shaded square, i.e. in $\mathbb{R}^N$.

6.3.2 Dimension reduction: the mean-variance framework

In order to solve explicitly the general allocation problem (6.62) we need to determine the functional dependence (6.63) of the index of satisfaction on all the moments and the dependence of each moment on the allocation.

Suppose that we can focus on the two first moments only and neglect all the higher moments. In other words, assume that (6.63) can be approximated as follows:

$$\mathcal{S}(\boldsymbol{\alpha}) \approx \widetilde{\mathcal{H}}\left(\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\}, \mathrm{Var}\left\{\Psi_{\boldsymbol{\alpha}}\right\}\right), \tag{6.67}$$

for a suitable bivariate function $\widetilde{\mathcal{H}}$. This approximation is quite satisfactory in a wide range of applications, see Section 6.5.1. In this case the general allocation problem (6.62) is much easier to solve.

Indeed, since all the indices of satisfaction $\mathcal{S}$ discussed in Chapter 5 are consistent with weak stochastic dominance, for a given level of variance of the objective, higher expected values of the objective are always appreciated, *no matter* the functional expression of $\widetilde{\mathcal{H}}$. Therefore, if for each target value of variance of the investor's objective we pursue its maximum possible expected value, we are guaranteed to capture the solution to the general allocation problem. In other words, the optimal allocation $\boldsymbol{\alpha}^*$ that solves (6.62) must belong to the one-parameter family $\boldsymbol{\alpha}(v)$ defined as follows:

$$\boldsymbol{\alpha}(v) \equiv \underset{\substack{\boldsymbol{\alpha} \in \mathcal{C} \\ \operatorname{Var}\{\Psi_{\boldsymbol{\alpha}}\} = v}}{\operatorname{argmax}} \operatorname{E}\{\Psi_{\boldsymbol{\alpha}}\}, \tag{6.68}$$

where $v \geq 0$. The optimization problem (6.68) is the *mean-variance* approach pioneered by Markowitz, see Markowitz (1991). Its solution is called the *mean-variance efficient frontier*.

Therefore the general problem (6.62) is reduced to a two-step recipe. The first step is the computation of the mean-variance efficient frontier, which can be performed easily as described in Section 6.3.3. The second step is the following one-dimensional search:

$$\boldsymbol{\alpha}^* \equiv \boldsymbol{\alpha}(v^*) \equiv \underset{v \geq 0}{\operatorname{argmax}} \, \mathcal{S}(\boldsymbol{\alpha}(v)), \tag{6.69}$$

which can be computed numerically when analytical results are not available, see the case study in Section 6.7.

The mean-variance approach appeals intuition. The target variance v of the investor's objective $\Psi_{\boldsymbol{\alpha}}$ in the mean-variance optimization (6.68) can be interpreted as the riskiness of the solution $\boldsymbol{\alpha}(v)$: for a given level of risk v, the investor seeks the allocation that maximizes the expected value of his objective. As the risk level v spans all the positive numbers, the one-parameter family of solutions $\boldsymbol{\alpha}(v)$ describes a one-dimensional curve in the N-dimensional space of all possible allocations, and the optimal allocation $\boldsymbol{\alpha}^*$ must lie on this curve.

Making use of the Lagrangian formulation in (6.68), we can express the optimal allocation (6.69) as follows:

$$\boldsymbol{\alpha}^*, \lambda^* \equiv \underset{\boldsymbol{\alpha} \in \mathcal{C}, \lambda \in \mathbb{R}}{\operatorname{argmax}} \{\operatorname{E}\{\Psi_{\boldsymbol{\alpha}}\} - \lambda (\operatorname{Var}\{\Psi_{\boldsymbol{\alpha}}\} - v^*)\}. \tag{6.70}$$

The Lagrange coefficient λ^* that solves (6.70) can be interpreted as a coefficient of risk aversion. If λ^* is null the investor is risk neutral: indeed, the argument in curly brackets in (6.70) becomes the expected value. Thus the risk premium required by the investor to be exposed to market risk is null, see (5.89). On the other hand, if λ^* is positive the investor is risk averse: indeed, allocations with the same expected value but with larger variance are penalized in (6.70). Similarly, if λ^* is negative the investor is risk prone.

6.3.3 Setting up the mean-variance optimization

We recall that the investor's objective that appears in the mean-variance problem (6.68) is a linear function of the allocation and of the market vector:

$$\Psi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\mathbf{M}, \tag{6.71}$$

see (6.2). Using the affine equivariance (2.56) and (2.71) of the expected value and the covariance respectively we obtain:

$$\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} = \boldsymbol{\alpha}' \,\mathrm{E}\{\mathbf{M}\} \tag{6.72}$$

$$\mathrm{Var}\{\Psi_{\boldsymbol{\alpha}}\} = \boldsymbol{\alpha}' \,\mathrm{Cov}\{\mathbf{M}\}\, \boldsymbol{\alpha}. \tag{6.73}$$

Therefore we can re-express the mean-variance efficient frontier (6.68) in the following form:

$$\boldsymbol{\alpha}(v) \equiv \underset{\substack{\boldsymbol{\alpha}\in\mathcal{C} \\ \boldsymbol{\alpha}'\,\mathrm{Cov}\{\mathbf{M}\}\boldsymbol{\alpha}=v}}{\operatorname{argmax}} \quad \boldsymbol{\alpha}' \,\mathrm{E}\{\mathbf{M}\}, \tag{6.74}$$

where $v \geq 0$.

In addition to the set of constraints $\mathcal{C}$, the only inputs required to compute the mean-variance efficient frontier (6.74) are the expected values of the market vector $\mathrm{E}\{\mathbf{M}\}$ and the respective covariance matrix $\mathrm{Cov}\{\mathbf{M}\}$. In order to compute these inputs, we have to follow the steps below, adapting from the discussion in Section 6.1:

Step 1. Detect the invariants $\mathbf{X}_{t,\widetilde{\tau}}$ behind the market relative to a suitable estimation horizon $\widetilde{\tau}$, see Section 3.1.

Step 2. Estimate the distribution of the invariants $\mathbf{X}_{t,\widetilde{\tau}}$, see Chapter 4.

Step 3. Project the invariants $\mathbf{X}_{t,\widetilde{\tau}}$ to the investment horizon, obtaining the distribution of $\mathbf{X}_{T+\tau,\tau}$, see Section 3.2.

Step 4. Map the distribution of the invariants $\mathbf{X}_{T+\tau,\tau}$ into the distribution of the prices at the investment horizon of the securities $\mathbf{P}_{T+\tau}$, see Section 3.3.

Step 5. Compute the expected value $\mathrm{E}\{\mathbf{P}_{T+\tau}\}$ and the covariance matrix $\mathrm{Cov}\{\mathbf{P}_{T+\tau}\}$ of the distribution of the market prices.

Step 6. Compute the inputs for the optimization (6.74), i.e. the expected value and the covariance matrix of the market vector $\mathbf{M}$. The market vector is an affine transformation of the market prices $\mathbf{M} \equiv \mathbf{a} + \mathbf{B}\mathbf{P}_{T+\tau}$, see (6.3). Therefore the inputs of the optimization follow from the affine equivariance (2.56) and (2.71) of the expected value and of the covariance matrix respectively:

$$\mathrm{E}\{\mathbf{M}\} = \mathbf{a} + \mathbf{B}\,\mathrm{E}\{\mathbf{P}_{T+\tau}\} \tag{6.75}$$

$$\mathrm{Cov}\{\mathbf{M}\} = \mathbf{B}\,\mathrm{Cov}\{\mathbf{P}_{T+\tau}\}\,\mathbf{B}'. \tag{6.76}$$

If the market is composed of equity-like and fixed-income security without derivative products, we can bypass some of the above steps. Indeed, in this case the invariants $\mathbf{X}_{t,\widetilde{\tau}}$ are the compounded returns and the changes in yield to maturity respectively. From (3.100) we obtain the expected value of the prices $\mathbf{P}_{T+\tau}$ directly from the distribution of the market invariants relative to the estimation interval:

$$\mathrm{E}\left\{P_{T+\tau}^{(n)}\right\} = e^{\boldsymbol{\gamma}'\boldsymbol{\delta}^{(n)}} \left[\phi_{\mathbf{X}_{t,\widetilde{\tau}}}\left(-i\,\mathrm{diag}\left(\boldsymbol{\varepsilon}\right)\boldsymbol{\delta}^{(n)}\right)\right]^{\frac{\tau}{\widetilde{\tau}}}, \tag{6.77}$$

where ϕ is the characteristic function of the market invariants, $\boldsymbol{\gamma}$ and $\boldsymbol{\varepsilon}$ are constant vectors defined in (3.84) and (3.85), and $\boldsymbol{\delta}$ is the canonical basis (A.15). Similarly, form (3.100) we obtain:

$$\mathrm{E}\left\{P_{T+\tau}^{(n)} P_{T+\tau}^{(m)}\right\} = e^{\boldsymbol{\gamma}'\left(\boldsymbol{\delta}^{(n)}+\boldsymbol{\delta}^{(m)}\right)} \left[\phi_{\mathbf{X}_{t,\tilde{\tau}}}\left(-i\,\mathrm{diag}\,(\boldsymbol{\varepsilon})\left(\boldsymbol{\delta}^{(n)}+\boldsymbol{\delta}^{(m)}\right)\right)\right]^{\frac{\tau}{\tilde{\tau}}}. \tag{6.78}$$

This expression with (6.77) in turn yields the covariance matrix:

$$\mathrm{Cov}\left\{P_{T+\tau}^{(n)}, P_{T+\tau}^{(m)}\right\} = \mathrm{E}\left\{P_{T+\tau}^{(n)} P_{T+\tau}^{(m)}\right\} - \mathrm{E}\left\{P_{T+\tau}^{(n)}\right\}\mathrm{E}\left\{P_{T+\tau}^{(m)}\right\}. \tag{6.79}$$

The inputs for the optimization (6.74) then follow from (6.75) and (6.76). For an application of these formulas, see the case study in Section 6.7.

If the constraints $\mathcal{C}$ in (6.74) are not too complex the computation of the mean-variance efficient frontier represents a quadratic programming problem which can be easily solved numerically, see Section 6.5.3.

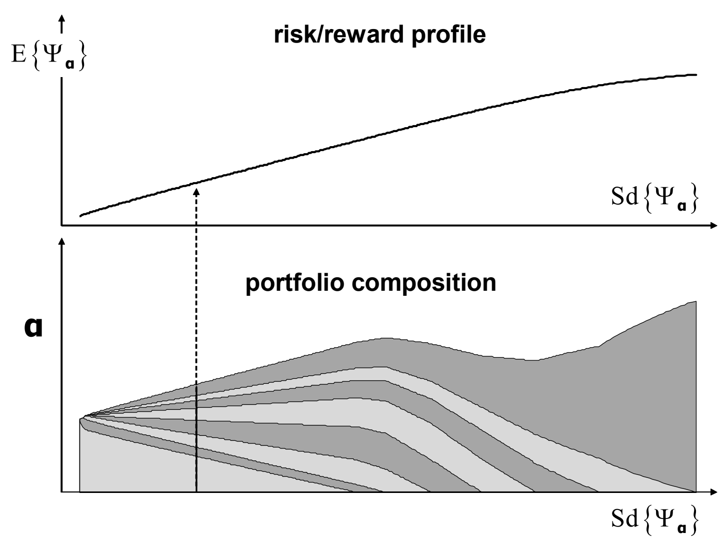

Fig. 6.7. Mean-variance efficient frontier

In Figure 6.7 we computed the solutions for the standard problem where the investor is bound by a budget constraint and a no-short-sale constraint:

$$\mathcal{C}: \ \boldsymbol{\alpha}'\mathbf{p}_T = w_T, \, \boldsymbol{\alpha} \geq \mathbf{0}. \tag{6.80}$$

Refer to `symmys.com` for the details on the market and on the computations.

If the constraints are affine, it is even possible to compute the analytical solution of the mean-variance problem, see Section 6.4 for the theory and Section 6.6 for an application.

6.3.4 Mean-variance in terms of returns

Recall from (3.10) that the linear return from the investment date T to the investment horizon τ of a security/portfolio that at time t trades at the price P_t is defined as follows:

$$L \equiv \frac{P_{T+\tau}}{P_T} - 1. \tag{6.81}$$

The mean-variance approach (6.74) is often presented and solved in terms of the returns (6.81) instead of the market vector as in (6.74). Nevertheless, this formulation presents a few drawbacks.

To present the formulation in terms of returns we need to make two restrictive assumptions. First, we assume that the investor's objective is final wealth, or equivalently that the market vector in (6.71) is represented by the prices of the securities at the investment horizon:

$$\Psi_\alpha \equiv \boldsymbol{\alpha}'\mathbf{P}_{T+\tau}. \tag{6.82}$$

Second, we assume that the investor's initial capital is not null:

$$w_T \equiv \boldsymbol{\alpha}'\mathbf{p}_T \neq 0. \tag{6.83}$$

Consider the linear return on wealth:

$$L^{\Psi_\alpha} \equiv \frac{\Psi_\alpha}{w_T} - 1. \tag{6.84}$$

As we show in Appendix www.6.6, under the assumptions (6.82) and (6.83) the mean-variance efficient frontier (6.68) can be expressed equivalently in terms of the linear return on wealth as follows:

$$\boldsymbol{\alpha}(v) = \underset{\substack{\boldsymbol{\alpha} \in \mathcal{C} \\ \mathrm{Var}\left\{L^{\Psi_\alpha}\right\} = v}}{\operatorname{argmax}} \mathrm{E}\left\{L^{\Psi_\alpha}\right\}, \tag{6.85}$$

where $v \geq 0$.

Consider now the *relative weights* $\mathbf{w}$ of a generic allocation:

$$\mathbf{w} \equiv \frac{\mathrm{diag}(\mathbf{p}_T)}{\boldsymbol{\alpha}'\mathbf{p}_T} \boldsymbol{\alpha}. \tag{6.86}$$

Since the current prices $\mathbf{p}_T$ are known, the relative weights $\mathbf{w}$ are a scale-independent equivalent representation of the allocation $\boldsymbol{\alpha}$.

As we show in Appendix www.6.6, we can express the linear return on wealth in terms of the linear returns (6.81) of the securities in the market and the respective relative weights:

$$L^{\Psi_\alpha} = \mathbf{w}'\mathbf{L}. \tag{6.87}$$

Therefore, using the affine equivariance properties (2.56) and (2.71) of the expected value and of the covariance matrix respectively, we can write (6.85) equivalently as follows:

$$\mathbf{w}(v) = \underset{\substack{\mathbf{w}\in\mathcal{C} \\ \mathbf{w}'\operatorname{Cov}\{\mathbf{L}\}\mathbf{w}=v}}{\operatorname{argmax}} \mathbf{w}'\operatorname{E}\{\mathbf{L}\}, \tag{6.88}$$

where $v \geq 0$. The efficient frontier in terms of the allocation vector $\boldsymbol{\alpha}(v)$ is then recovered from the relative weights (6.88) by inverting (6.86).

In order to set up the optimization in terms of linear returns and relative weights (6.88) we proceed like in the more general mean-variance case (6.74). Indeed, the inputs necessary to solve (6.88) are the expected value of the horizon-specific linear returns $\operatorname{E}\{\mathbf{L}\}$ and the respective covariance matrix $\operatorname{Cov}\{\mathbf{L}\}$. These parameters are obtained by following steps similar to those on p. 321:

Step 1. Detect the invariants $\mathbf{X}_{t,\widetilde{\tau}}$ behind the market relative to a suitable estimation horizon $\widetilde{\tau}$, see Section 3.1.

Step 2. Estimate the distribution of the invariants $\mathbf{X}_{t,\widetilde{\tau}}$, see Chapter 4.

Step 3. Project these invariants $\mathbf{X}_{t,\widetilde{\tau}}$ to the investment horizon, obtaining the distribution of $\mathbf{X}_{T+\tau,\tau}$, see Section 3.2.

Step 4. Map the distribution of the invariants $\mathbf{X}_{T+\tau,\tau}$ into the distribution of the prices at the investment horizon of the securities $\mathbf{P}_{T+\tau}$, see Section 3.3.

Step 5. Compute the expected value $\operatorname{E}\{\mathbf{P}_{T+\tau}\}$ and the covariance matrix $\operatorname{Cov}\{\mathbf{P}_{T+\tau}\}$ from the distribution of the market prices.

Step 6. Compute the inputs for the optimization (6.74), i.e. the expected value and the covariance matrix of the linear returns, from (6.81) using the affine equivariance (2.56) and (2.71) of the expected value and of the covariance matrix respectively:

$$\operatorname{E}\{\mathbf{L}\} = \operatorname{diag}(\mathbf{p}_T)^{-1}\operatorname{E}\{\mathbf{P}_{T+\tau}\} - \mathbf{1} \tag{6.89}$$

$$\operatorname{Cov}\{\mathbf{L}\} = \operatorname{diag}(\mathbf{p}_T)^{-1}\operatorname{Cov}\{\mathbf{P}_{T+\tau}\}\operatorname{diag}(\mathbf{p}_T)^{-1}. \tag{6.90}$$

If the constraints $\mathcal{C}$ in (6.88) are not too complex, the optimization problem in terms of linear returns and relative weights is quadratic and therefore it can be solved easily either analytically or numerically, just like the more general problem (6.74).

On the other hand, expressing an allocation in terms relative weights is somewhat more intuitive than expressing it in absolute terms. In other words, it is easier to interpret a statement such as "thirty percent of one's budget is invested in xyz" than "his investment consists, among others, of a thousand shares of xyz". Furthermore, in the formulation in terms of linear returns and relative weights a few expressions assume a simpler form.

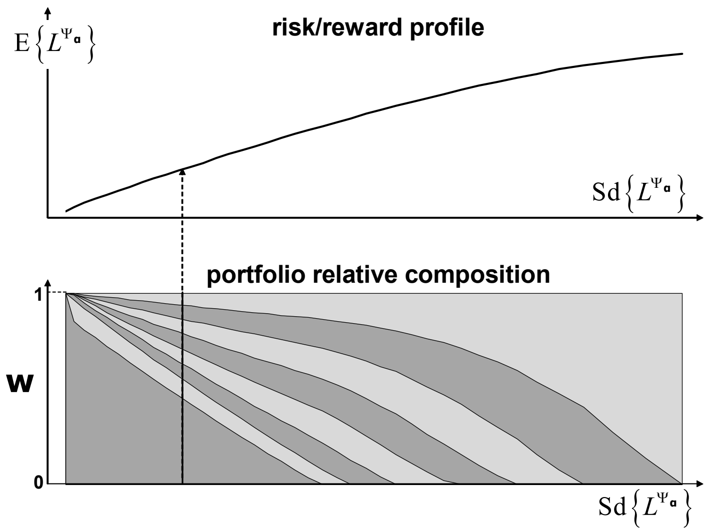

Fig. 6.8. MV efficient frontier in terms of returns and relative weights

For instance, the standard problem where the investor is bound by a budget constraint and a no-short-sale constraint as in (6.80) simplifies as follows:

$$\mathcal{C}: \ \mathbf{w}'\mathbf{1} = 1, \ \mathbf{w} \geq \mathbf{0}. \tag{6.91}$$

In Figure 6.8 we computed the efficient frontier under these constraints: compare with the respective plots in Figure 6.7. Refer to `symmys.com` for the details on the market and on the computations.

For the above reasons, the mean-variance framework is often presented in terms of returns and relative weights.

Nevertheless, we stress that the specification in terms of returns is not as general as the specification in terms of the investor's objective, because it applies only under the hypotheses (6.82) and (6.83). For instance, the linear returns on wealth are not defined when the initial investment is null. This prevents the analysis of *market-neutral strategies*, namely highly leveraged portfolios that pursue the largest possible final wealth by allocating zero initial net capital.

Furthermore, the formulation of the mean-variance problem in terms of returns and relative weights gives rise to misunderstandings. Indeed it makes it harder to separate the estimation process from the optimization process, see Section 6.5.4, and it gives rise to confusion when implementing allocation at different horizons, see Section 6.5.5.

6.4 Analytical solutions of the mean-variance problem

In Section 6.3.3 we set up the general mean-variance optimization of the investor's objective:

$$\boldsymbol{\alpha}(v) \equiv \underset{\substack{\boldsymbol{\alpha} \in \mathcal{C} \\ \boldsymbol{\alpha}' \operatorname{Cov}\{\mathbf{M}\} \boldsymbol{\alpha} = v}}{\operatorname{argmax}} \boldsymbol{\alpha}' \operatorname{E}\{\mathbf{M}\}, \tag{6.92}$$

where $v \geq 0$, and we discussed the steps necessary to compute the inputs of this problem, namely the expected values of the market vector $\operatorname{E}\{\mathbf{M}\}$ and the respective covariance matrix $\operatorname{Cov}\{\mathbf{M}\}$.

In this section we assume knowledge of these inputs and we analyze the explicit solution of the mean-variance optimization assuming that the constraints in (6.92) are affine:

$$\mathcal{C}: \ \mathbf{D}\boldsymbol{\alpha} = \mathbf{c}, \tag{6.93}$$

where $\mathbf{D}$ is a full-rank $K \times N$ matrix whose rows are not collinear with the expectation on the market $\operatorname{E}\{\mathbf{M}\}$, and $\mathbf{c}$ is a K-dimensional vector. When the constraints are affine the mean-variance efficient allocations (6.92) can be computed analytically. The analytical solution provides insight into the effect of the constraints and of the market parameters on the efficient frontier and on the investor's satisfaction in more general situations.

In particular, we focus one affine constraint. In other words (6.93) becomes:

$$\mathcal{C}: \ \mathbf{d}'\boldsymbol{\alpha} = c, \tag{6.94}$$

where $\mathbf{d}$ is a generic constant vector not collinear with the expectation on the market $\operatorname{E}\{\mathbf{M}\}$ and c is a scalar. The one-dimensional case is still general enough to cover a variety of practical situations. Furthermore, in the one-dimensional case the analytical solution is very intuitive and easy to interpret geometrically. The computations and respective interpretations for the general case (6.93) follow similarly to the one-dimensional case.

The most notable example of affine constraint is the budget constraint:

$$\mathcal{C}: \ \boldsymbol{\alpha}'\mathbf{p}_T = w_T, \tag{6.95}$$

where w_T is the investor's initial capital.

6.4.1 Efficient frontier with affine constraints

In the general case where $c \neq 0$ in (6.94), the mean-variance efficient frontier (6.92) is the set of non-empty solutions to this problem:

$$\boldsymbol{\alpha}(v) \equiv \underset{\substack{\boldsymbol{\alpha}'\mathbf{d}=c \\ \mathrm{Var}\{\Psi_{\boldsymbol{\alpha}}\}=v}}{\operatorname{argmax}} \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\}, \tag{6.96}$$

where $v \geq 0$.

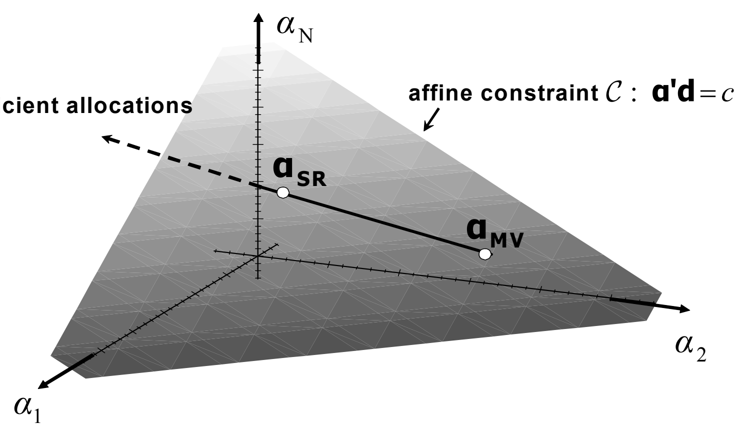

Fig. 6.9. MV efficient allocations under affine constraint: two-fund separation

In Appendix www.6.3 we prove that the above solutions are more easily parametrized in terms of the expected value of the investor's objective $e \equiv \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\}$ and read explicitly:

$$\boldsymbol{\alpha}(e) = \boldsymbol{\alpha}_{MV} + [e - \mathrm{E}\{\Psi_{\boldsymbol{\alpha}_{MV}}\}] \frac{\boldsymbol{\alpha}_{SR} - \boldsymbol{\alpha}_{MV}}{\mathrm{E}\{\Psi_{\boldsymbol{\alpha}_{SR}}\} - \mathrm{E}\{\Psi_{\boldsymbol{\alpha}_{MV}}\}}. \tag{6.97}$$

In this expression the scalar e varies in an infinite range:

$$e \in [\mathrm{E}\{\Psi_{\boldsymbol{\alpha}_{MV}}\}, \infty); \tag{6.98}$$

and the two allocations $\boldsymbol{\alpha}_{SR}$ and $\boldsymbol{\alpha}_{MV}$ are defined as follows:

$$\boldsymbol{\alpha}_{MV} \equiv \frac{c \,\mathrm{Cov}\{\mathbf{M}\}^{-1} \mathbf{d}}{\mathbf{d}' \,\mathrm{Cov}\{\mathbf{M}\}^{-1} \mathbf{d}}. \tag{6.99}$$

$$\boldsymbol{\alpha}_{SR} \equiv \frac{c \,\mathrm{Cov}\{\mathbf{M}\}^{-1} \mathrm{E}\{\mathbf{M}\}}{\mathbf{d}' \,\mathrm{Cov}\{\mathbf{M}\}^{-1} \mathrm{E}\{\mathbf{M}\}}. \tag{6.100}$$

In other words, the mean-variance efficient frontier (6.97) is a straight semi-line in the N-dimensional space of allocations that lies on the $(N-1)$-dimensional hyperplane determined by the affine constraint. This straight semi-line stems from the allocation $\boldsymbol{\alpha}_{MV}$ and passes through the allocation $\boldsymbol{\alpha}_{SR}$, see Figure 6.9.

This result is known as the *two-fund separation theorem*: a linear combination of two specific portfolios (mutual funds) suffices to generate the whole mean-variance efficient frontier.

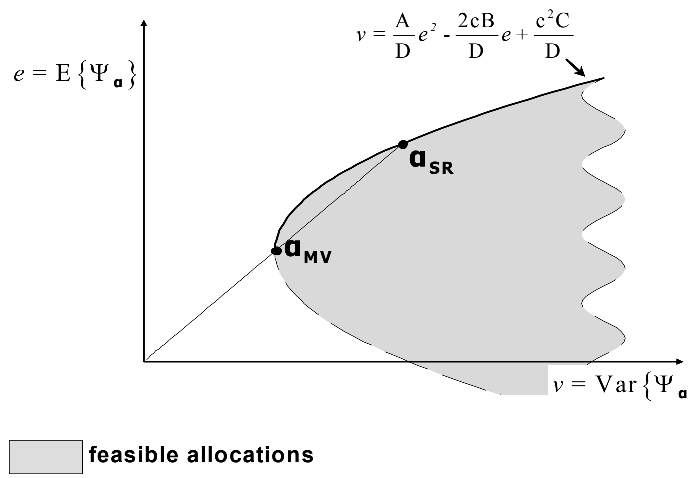

Fig. 6.10. Risk/reward profile of MV efficient allocations: expected value and variance

To evaluate the investor's satisfaction ensuing from the efficient allocations, we recall that in the mean-variance setting the investor's satisfaction by assumption only depends on the expected value and the variance of the investor's objective, see (6.67). Therefore we consider the plane of these two moments:

$$(v, e) \equiv (\mathrm{Var}\{\Psi\}, \mathrm{E}\{\Psi\}) . \tag{6.101}$$

In Appendix www.6.3 we show that the feasible set in these coordinates is the region to the right of the following parabola, see Figure 6.10:

$$v = \frac{A}{D}e^2 - \frac{2cB}{D}e + \frac{c^2C}{D}, \tag{6.102}$$

where (A, B, C, D) are four scalars that do not depend on the allocations:

$$\begin{array}{ll} A \equiv \mathbf{d}' \mathrm{Cov}\{\mathbf{M}\}^{-1} \mathbf{d} & B \equiv \mathbf{d}' \mathrm{Cov}\{\mathbf{M}\}^{-1} \mathrm{E}\{\mathbf{M}\} \\ C \equiv \mathrm{E}\{\mathbf{M}\}' \mathrm{Cov}\{\mathbf{M}\}^{-1} \mathrm{E}\{\mathbf{M}\} & D \equiv AC - B^2, \end{array} \tag{6.103}$$

From (6.96), the mean-variance efficient frontier (6.97) corresponds to the allocations that give rise to the upper branch of this parabola.

The allocations $\boldsymbol{\alpha}_{SR}$ and $\boldsymbol{\alpha}_{MV}$ that generate the efficient frontier (6.97) are very special in terms of their risk/reward profile.

As we show in Appendix www.6.3, $\boldsymbol{\alpha}_{MV}$ is the allocation that displays the least possible variance. Therefore $\boldsymbol{\alpha}_{MV}$ is called the *global minimum variance portfolio*: in the risk/reward plane of Figure 6.10 the allocation $\boldsymbol{\alpha}_{MV}$ corresponds to the "belly" of the parabola (6.102).

To interpret the allocation $\boldsymbol{\alpha}_{SR}$, we recall that the Sharpe ratio (5.51) is defined as the ratio of the expected value of the investor's objective over its standard deviation:

$$\mathrm{SR}(\boldsymbol{\alpha}) \equiv \frac{\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\}}{\mathrm{Sd}\{\Psi_{\boldsymbol{\alpha}}\}}. \tag{6.104}$$

As we show in Appendix www.6.3, $\boldsymbol{\alpha}_{SR}$ is the allocation that displays the highest possible Sharpe ratio. Therefore $\boldsymbol{\alpha}_{SR}$ is called the *maximum Sharpe ratio portfolio*: in the risk/reward plane of Figure 6.10 the allocation $\boldsymbol{\alpha}_{SR}$ represents the intersection of the efficient frontier with the straight line through the origin and the minimum variance portfolio, see Appendix www.6.3.

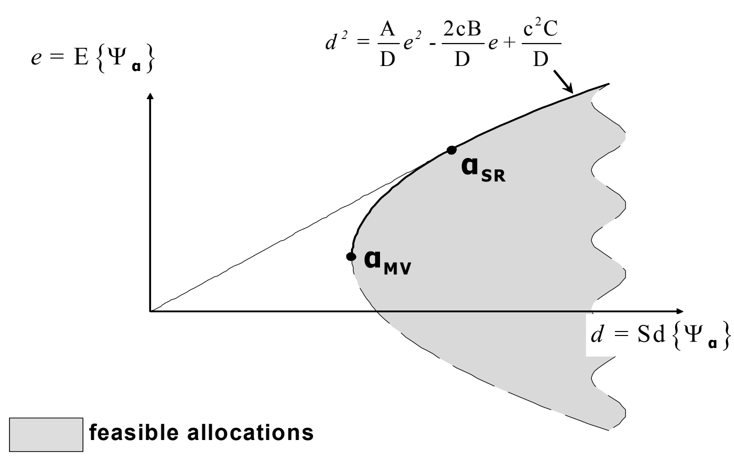

Fig. 6.11. Risk/reward profile of MV efficient allocations: expected value and standard deviation

Due to the interpretation in terms of the Sharpe ratio, it is convenient to represent the risk/reward profile of the objective also in terms of the expected value and standard deviation:

$$(d, e) \equiv (\mathrm{Sd}\{\Psi\}, \mathrm{E}\{\Psi\}). \tag{6.105}$$

In this plane the boundary of the feasible set, namely the parabola (6.102), becomes the following hyperbola, see Figure 6.11:

$$d^2 = \frac{A}{D}e^2 - \frac{2cB}{D}e + \frac{c^2 C}{D}, \quad d > 0. \tag{6.106}$$

In turn, the global minimum variance portfolio is also the global minimum standard-deviation portfolio and therefore it plots as the "belly" of the hyperbola. On the other hand, from the definition of the Sharpe ratio (6.104), the maximum Sharpe ratio portfolio corresponds to the point of tangency of the hyperbola with a straight line stemming from the origin.

6.4.2 Efficient frontier with linear constraints

When $c \equiv 0$ in the affine constraint (6.94), the constraint becomes linear and the mean-variance efficient frontier (6.92) becomes the set of non-empty solutions to this problem:

$$\boldsymbol{\alpha}(v) \equiv \underset{\substack{\boldsymbol{\alpha}'\mathbf{d}=0 \\ \operatorname{Var}\{\Psi_{\boldsymbol{\alpha}}\}=v}}{\operatorname{argmax}} \operatorname{E}\{\Psi_{\boldsymbol{\alpha}}\}, \tag{6.107}$$

where $v \geq 0$. This special case recurs in many applications.

A notable example is provided by market-neutral strategies which invest with infinite leverage: by selling short some securities one can finance the purchase of other securities and thus set up positions that have zero initial value:

$$\mathcal{C}: \ \boldsymbol{\alpha}'\mathbf{p}_T = 0. \tag{6.108}$$

Another important example is provided by allocations against a benchmark, which we discuss extensively in Section 6.6.

In Appendix www.6.3 we prove that the above solutions are more easily parametrized in terms of the expected value of the investor's objective $e \equiv \operatorname{E}\{\Psi_{\boldsymbol{\alpha}}\}$ and read explicitly:

$$\boldsymbol{\alpha}(e) = e\boldsymbol{\alpha}_0, \tag{6.109}$$

where $\boldsymbol{\alpha}_0$ is a specific fixed allocation, defined in terms of the constants (6.103) as follows:

$$\boldsymbol{\alpha}_0 \equiv \operatorname{Cov}\{\mathbf{M}\}^{-1}\left(A\operatorname{E}\{\mathbf{M}\} - B\mathbf{d}\right). \tag{6.110}$$

In other words, when the investment constraint is linear, the ensuing mean-variance efficient allocations (6.109) describe a straight semi-line stemming from the origin that passes through the specific allocation $\boldsymbol{\alpha}_0$ and lies on the $(N-1)$-dimensional hyperplane determined by the constraint $\boldsymbol{\alpha}'\mathbf{d} = 0$, see Figure 6.12.

This result can be seen as a special case of (6.97) and Figure 6.9 in the limit where the constant c in (6.96) tends to zero. Indeed, in this limit the

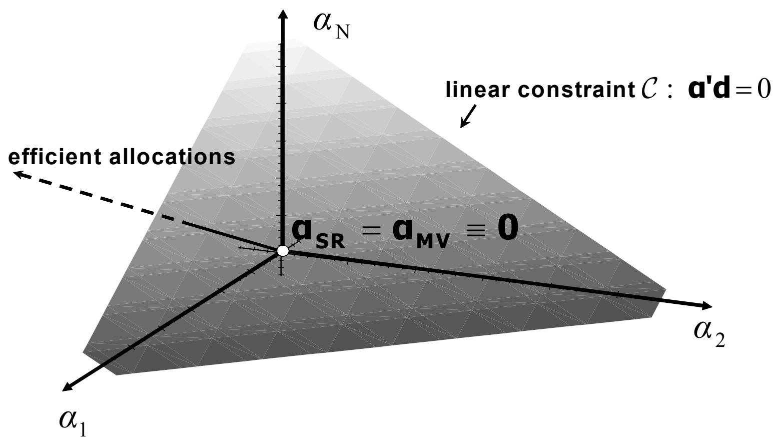

Fig. 6.12. MV efficient allocations under linear constraint

two portfolios $\boldsymbol{\alpha}_{SR}$ and $\boldsymbol{\alpha}_{MV}$, defined in (6.99) and (6.100) respectively, both shrink to zero. Nevertheless, the direction of departure of the straight semi-line (6.97) from the global minimum-variance portfolio does not depend on c and thus remains constant:

$$\boldsymbol{\alpha}_0 = \frac{\boldsymbol{\alpha}_{SR} - \boldsymbol{\alpha}_{MV}}{\mathrm{E}\{\Psi_{\boldsymbol{\alpha}_{SR}}\} - \mathrm{E}\{\Psi_{\boldsymbol{\alpha}_{MV}}\}}. \tag{6.111}$$

Therefore, as $c \to 0$ in the constraint (6.94), the straight semi-line in Figure 6.9 shifts in a parallel way towards the origin.

As in the case of a generic affine constraint, also for the special case $c \equiv 0$ in order to analyze the satisfaction ensuing from the mean-variance efficient allocations we only need to focus on the first two moments of the objective (6.101), or equivalently (6.105). In the latter coordinates, the hyperbola (6.106) which limits the feasible set in Figure 6.11 degenerates into the following locus:

$$e = \pm\sqrt{\frac{D}{A}}d, \quad d \geq 0. \tag{6.112}$$

This locus represents two straight semi-lines that stem from the origin, see Figure 6.13. The efficient frontier corresponds to the upper branch of this degenerate hyperbola, i.e. the straight line in the positive quadrant. Therefore all the efficient allocations share the same Sharpe ratio which is the highest possible in the feasible set and is equal to $\sqrt{D/A}$.

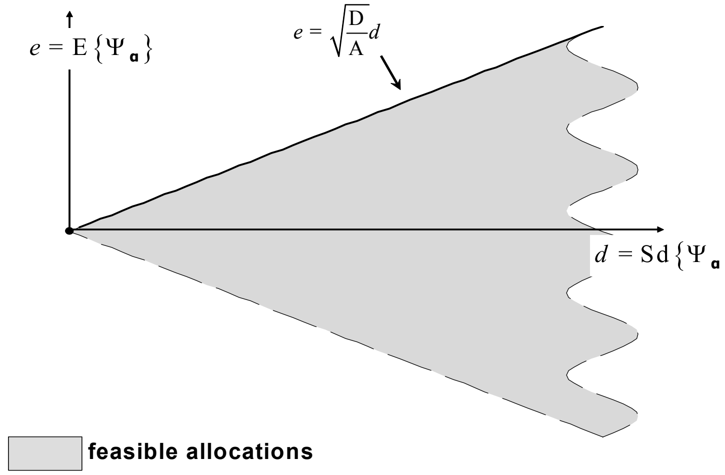

Fig. 6.13. Risk/reward profile of MV efficient allocations under linear constraint

6.4.3 Effects of correlations and other parameters

The only market parameters necessary to determine the mean-variance efficient frontier (6.92) are the expected values of the market vector $\mathrm{E}\{\mathbf{M}\}$ and the covariance matrix $\mathrm{Cov}\{\mathbf{M}\}$, which we factor into the respective standard deviations and correlations:

$$\mathrm{Cov}\{\mathbf{M}\} \equiv \mathrm{diag}(\mathrm{Sd}\{\mathbf{M}\})\,\mathrm{Cor}\{\mathbf{M}\}\,\mathrm{diag}(\mathrm{Sd}\{\mathbf{M}\}). \tag{6.113}$$

In this section we discuss the impact of changes in these parameters on the investor's satisfaction.

In the mean-variance setting the investor's satisfaction only depends on the expected value and the variance of his objective, see (6.67), or equivalently on the expected value and the standard deviation of his objective. Therefore we analyze the effects of changes in the market parameters in the plane of these coordinates:

$$(d, e) \equiv (\mathrm{Sd}\{\Psi\}, \mathrm{E}\{\Psi\}). \tag{6.114}$$

Since all the indices of satisfaction $\mathcal{S}$ discussed in Chapter 5 are consistent with weak stochastic dominance, for a given level of standard deviation of the objective, higher expected values of the objective are always appreciated. Therefore a given market presents better investment opportunities than another market if, other things equal, the upper boundary of its feasible set in the coordinates (6.114) plots above the upper boundary of the feasible set of the other market for all values of the standard deviation.

It is immediate to determine the effect of changes in expected values $\mathrm{E}\{\mathbf{M}\}$ and standard deviations $\mathrm{Sd}\{\mathbf{M}\}$ on the feasible set. Indeed, larger expected values of the market vector shift the feasible set upward in the coordinates (6.114) and larger standard deviations shift the feasible set to the right.

To analyze the effect of the correlations, we consider the simplest case of a two-security market, which gives rise to a bivariate market vector $\mathbf{M} \equiv (M_1, M_2)'$. In this case there exists only one correlation $\rho \equiv \operatorname{Cor}\{M_1, M_2\}$.

We consider the generic case where with a higher expected value of the investor's objective is associated a higher standard deviation. Therefore we assume without loss of generality:

$$e^{(1)} < e^{(2)}, \qquad d^{(1)} < d^{(2)}, \tag{6.115}$$

where the index $j = 1, 2$ denotes the coordinates (6.114) of a full allocation in the j-th security. The boundary of the feasible set (6.106) becomes fully determined by the value of the correlation ρ. In Figure 6.14 we show the effect of different values of the market correlation ρ on the feasible set, see Appendix www.6.4 for the analytical expressions behind these plots and the statements that follow.

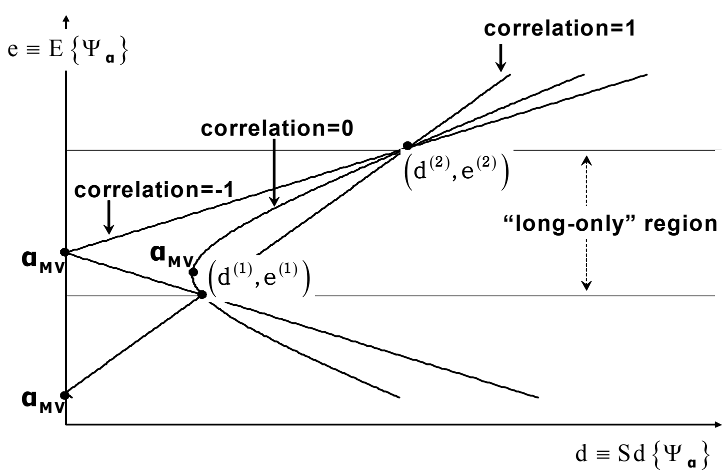

Fig. 6.14. Diversification effect of correlation

We distinguish three cases for the correlation: total correlation $\rho \equiv 1$, null correlation, $\rho \equiv 0$ and total anti-correlation $\rho \equiv -1$; and two cases for the allocation: *long-only positions*, where the amounts (α_1, α_2) of both securities in an allocation are positive, and *short positions*, where one of the two amounts α_1 or α_2 is negative.

In the case of perfect positive correlation the two securities are perceived as equivalent. The efficient frontier degenerates into a straight line that joins the coordinates of the two assets in the plane (6.114):

$$\rho \equiv 1 \quad \Rightarrow \quad e = e^{(1)} + \left(d - d^{(1)}\right) \frac{e^{(2)} - e^{(1)}}{d^{(2)} - d^{(1)}}. \tag{6.116}$$

By shorting one of the assets it is possible to completely hedge the risk of the other asset and achieve a global minimum-variance portfolio $\boldsymbol{\alpha}_{MV}$ such that the investor's objective has null standard deviation. Nevertheless, the perfect hedge comes at a price: the expected value of the investor's objective delivered by the zero-variance allocation is worse than the expected value delivered by a full allocation in the asset with the lower expected value:

$$\rho \equiv 1 \quad \Rightarrow \quad \mathrm{Sd}\left\{\Psi_{\boldsymbol{\alpha}_{MV}}\right\} = 0,\ \mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}_{MV}}\right\} < e^{(1)}. \tag{6.117}$$

As the correlation decreases toward zero, the securities give rise to an increasingly diversified market: the diversification effect makes the expected value of the global minimum-variance portfolio to rise, although the variance of this portfolio is no longer zero. In the long-only region the efficient frontier swells upwards, providing better investment opportunities than the straight-line (6.116).

As the market become fully anti-correlated, the efficient frontier degenerates into another straight line:

$$\rho \equiv -1 \quad \Rightarrow \quad e = e^{(1)} + \left(d + d^{(1)}\right) \frac{e^{(2)} - e^{(1)}}{d^{(2)} + d^{(1)}}. \tag{6.118}$$

Like in the case of perfect positive correlation, also in this situation it is possible to completely hedge the risk of one asset with the other one, obtaining a minimum-variance portfolio $\boldsymbol{\alpha}_{MV}$ whose standard deviation is zero. Nevertheless, in the case of perfect negative correlation this can be achieved without shorting any of the securities. Furthermore, the expected value of the investor's objective delivered by the zero-variance allocation is better than the expected value delivered by a full allocation in the asset with the lower expected value:

$$\rho \equiv -1 \quad \Rightarrow \quad \mathrm{Sd}\left\{\Psi_{\boldsymbol{\alpha}_{MV}}\right\} = 0,\ \mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}_{MV}}\right\} > e^{(1)}. \tag{6.119}$$

A comparison of (6.117) and (6.119) shows the benefits of diversification for low-variance portfolios.

Nevertheless, the perfect-correlation efficient frontier (6.116) is steeper than the perfect-anticorrelation efficient frontier (6.118). The two lines intersect at the point $\left(d^{(2)}, e^{(2)}\right)$, which corresponds to a full-investment in the riskier asset. When the investor is willing to increase his risk in pursuit of higher expected values by abandoning the long-only region, the best opportunities are provided by a highly correlated market. Therefore, contrary to a common belief, markets with low correlations do not necessarily provide better investment opportunities.

6.4.4 Effects of the market dimension

So far we have assumed that the dimension N of the market is fixed. Suppose that we allow new securities in the market. As intuition suggests, the ensuing investment opportunities can only improve. Indeed, if for a given level of standard deviation of the investor's objective it is possible to achieve a determined expected value, we can obtain the same result with a larger set of assets by simply allocating zero wealth in the new securities.

Nevertheless, from the above discussion on the effect of diversification we can guess that enlarging the market not only does not worsen, but actually substantially improves the efficient frontier. In order to verify this ansatz, we consider a market of an increasing number of securities, where we screen the effect of correlations, variances and expected values.

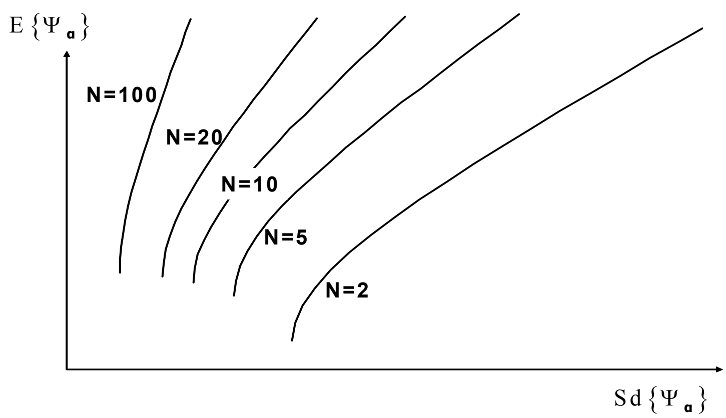

Fig. 6.15. Diversification effect of the dimension of the market

We consider a number N of assets whose expected values are equally spaced between two fixed extremes e_{lo} and e_{hi}:

$$\mathrm{E}\{\mathbf{M}\} \equiv (e_{lo}, e_{lo} + \Delta_N, \ldots, e_{hi} - \Delta_N, e_{hi})', \tag{6.120}$$

where $\Delta_N \equiv (e_{hi} - e_{lo}) / (N - 1)$. Similarly, we assume the standard deviations of these assets to be equally spaced between two fixed extremes d_{lo} and d_{hi}:

$$\mathrm{Sd}\{\mathbf{M}\} \equiv (d_{lo}, d_{lo} + \Gamma_N, \ldots, d_{hi} - \Gamma_N, d_{hi})', \tag{6.121}$$

where $\Gamma_N \equiv (d_{hi} - d_{lo}) / (N - 1)$. In order to screen out the effect of the cross-correlations, we assume zero correlation between all pairs of different entries of the market vector $\mathbf{M}$.

In Figure 6.15 we plot the efficient frontier in the above market as a function of the number N of securities. As expected, adding new assets shifts the frontier toward the upper-left region, giving rise to better investment opportunities: this effect is more pronounced when the number of assets in the market is relatively low.

6.5 Pitfalls of the mean-variance framework

In this section we discuss some common pitfalls in the interpretation and implementation of the mean-variance framework. Indeed, the very reasons that led to the success of the mean-variance approach also made it susceptible to misinterpretations.

6.5.1 MV as an approximation

We recall from (6.63) that the investor's satisfaction depends on all the moments of the distribution of the investor's objective:

$$\mathcal{S}(\boldsymbol{\alpha}) = \mathcal{H}\left(\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\}, \mathrm{CM}_2\left\{\Psi_{\boldsymbol{\alpha}}\right\}, \mathrm{CM}_3\left\{\Psi_{\boldsymbol{\alpha}}\right\}, \ldots\right), \tag{6.122}$$

where as in (6.2) the objective is a linear function of the allocation and of the market vector:

$$\Psi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\mathbf{M}. \tag{6.123}$$

The mean-variance approach relies on the approximation (6.67), according to which the investor's satisfaction is determined by the first two moments of the distribution of his objective:

$$\mathcal{S}(\boldsymbol{\alpha}) \approx \widetilde{\mathcal{H}}\left(\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\}, \mathrm{Var}\left\{\Psi_{\boldsymbol{\alpha}}\right\}\right), \tag{6.124}$$

where $\widetilde{\mathcal{H}}$ is a suitable bivariate function. This approximation is never exact. For this to be the case, the special conditions discussed below should apply to either the index of satisfaction $\mathcal{S}$ or to the distribution of the market $\mathbf{M}$.

The only index of satisfaction $\mathcal{S}$ such that the approximation (6.124) is exact no matter the market is the certainty-equivalent in the case of quadratic utility:

$$u(\psi) = \psi - \frac{1}{2\zeta}\psi^2. \tag{6.125}$$

Indeed in this case the expected utility becomes a function of the expected value and variance of the objective, and therefore so does the certainty-equivalent.

Nevertheless the quadratic utility is not flexible enough to model the whole spectrum of the investor's preferences. Furthermore, for values of the objective such that $\psi > \zeta$ the quadratic utility becomes nonsensical, as it violates the non-satiation principle underlying the investor's objective: larger values of the objective make the investor less satisfied, see (5.134) and comments thereafter.

The only markets such that the approximation (6.124) is exact no matter the index of satisfaction are elliptically distributed markets:

$$\mathbf{M} \sim \mathrm{El}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_N\right), \tag{6.126}$$

where $\boldsymbol{\mu}$ is the location parameter, $\boldsymbol{\Sigma}$ is the scatter matrix and g_N is the probability density generator for the N-dimensional case, see (2.268).

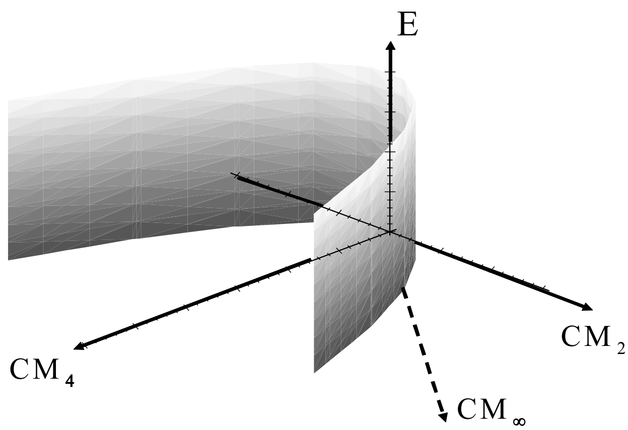

Fig. 6.16. Elliptical markets: the space of moments of the investor's objective is two-dimensional

Indeed, in this case from (2.270) and (2.276) the investor's objective is also elliptically distributed:

$$\Psi_{\boldsymbol{\alpha}} \sim \mathrm{El}\left(\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\}, \gamma \operatorname{Var}\left\{\Psi_{\boldsymbol{\alpha}}\right\}, g_1\right), \tag{6.127}$$

where g_1 is the pdf generator for the one-dimensional case and γ is a scalar that does not depend on the allocation. In other words, if the market $\mathbf{M}$ is elliptically distributed, the infinite-dimensional space of moments is reduced to a two-dimensional manifold parametrized by expected value and variance, see Figure 6.16. As a result, also the index of satisfaction (6.122) becomes a function of expected value and variance only.

Nevertheless, the assumption that a market is elliptical is very strong. For instance, in highly asymmetric markets with derivative products the elliptical assumption cannot be accepted. Even in the absence of derivatives, the standard distribution to model prices in the stock market is the multivariate lognormal distribution, which extends to a multivariate setting the classical framework of Black and Scholes (1973). If the stock market is very volatile or the investment horizon is large, approximating the lognormal distribution in the mean-variance problem with an elliptical distribution leads to incorrect results.

Although the approximation (6.124) is never exact, it is quite accurate in many practical applications, namely when the combined effects of the distribution of the market $\mathbf{M}$ and of the functional expression of $\mathcal{H}$ in (6.122) make the relative importance of higher moments negligible. Therefore, the applicability of the approximation (6.124) must be checked on a case-by-case basis.

6.5.2 MV as an index of satisfaction

Consider the mean-variance optimization (6.68), which we report here:

$$\boldsymbol{\alpha}(v) \equiv \underset{\substack{\boldsymbol{\alpha} \in \mathcal{C} \\ \mathrm{Var}\{\Psi_{\boldsymbol{\alpha}}\}=v}}{\operatorname{argmax}} \ \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\}, \tag{6.128}$$

where $v \geq 0$. To solve this problem we can parametrize the set of solutions $\boldsymbol{\alpha}(v)$ in terms of a Lagrange multiplier λ as follows:

$$\boldsymbol{\alpha}(\lambda) \equiv \underset{\boldsymbol{\alpha} \in \mathcal{C}}{\operatorname{argmax}} \{\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} - \lambda \,\mathrm{Var}\{\Psi_{\boldsymbol{\alpha}}\}\}, \tag{6.129}$$

where $\lambda \in \mathbb{R}$ can be interpreted as a level of risk aversion, see also (6.70) and comments thereafter.

From (6.69), the optimal allocation lies on the curve $\boldsymbol{\alpha}(\lambda)$. In other words, in order to determine the proper level of risk aversion λ^* and thus the optimal allocation $\boldsymbol{\alpha}^* \equiv \boldsymbol{\alpha}(\lambda^*)$, we perform the following one-dimensional optimization based on the investor's index of satisfaction:

$$\lambda^* \equiv \underset{\lambda \in \mathbb{R}}{\operatorname{argmax}} \, \mathcal{S}(\boldsymbol{\alpha}(\lambda)). \tag{6.130}$$

Consider an investor whose initial budget is one unit of currency. Assume that his objective is final wealth, and that he evaluates the riskiness of an allocation by means of a sensible index of satisfaction.

Suppose that the market consists of only two securities, that trade at the following price today:

$$p_T^{(1)} \equiv 1, \quad p_T^{(2)} \equiv 1. \tag{6.131}$$

At the investment horizon the value of the first security, which is non-stochastic, remains unaltered; the second security on the other hand has a 50% chance of doubling in value:

$$P_{T+\tau}^{(1)} \equiv 1, \quad P_{T+\tau}^{(2)} = \begin{cases} 1 \text{ (probability} = 50\%) \\ 2 \text{ (probability} = 50\%). \end{cases} \tag{6.132}$$

Taking into account the budget constraint and the no-short-sale constraint, the investor's objective is completely determined by the investment α in the risky security:

$$\Psi_{\alpha} \equiv (1-\alpha) P_{T+\tau}^{(1)} + \alpha P_{T+\tau}^{(2)}, \quad \alpha \in [0,1]. \tag{6.133}$$

From this expression and (6.132) it is immediate to compute the first two moments of the objective:

$$\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} = 1 + \frac{\alpha}{2}, \quad \mathrm{Var}\{\Psi_{\boldsymbol{\alpha}}\} = \frac{\alpha^2}{4}. \tag{6.134}$$

In turn, from the first-order condition in the Lagrange formulation (6.129) we obtain the mean-variance curve:

$$\alpha(\lambda) = \frac{1}{\lambda}. \tag{6.135}$$

To compute the optimal level λ^* that gives rise to the optimal allocation $\alpha(\lambda^*)$ we do not need to specify the investor's preferences by means of a specific index of satisfaction, as long as such an index is sensible. Indeed, sensibility implies a full investment in the risky security, which strongly dominates the risk-free asset. In other words, the optimal allocation is $\alpha^* \equiv 1$ and the respective optimal value for the Lagrange multiplier reads:

$$\lambda^* \equiv 1. \tag{6.136}$$

A common misinterpretation of the Lagrangian reformulation consists in considering the level of risk aversion λ^* as a feature of the investor that is independent of the market. In other words, one is tempted to first define a pseudo-index of satisfaction as follows:

$$\mathcal{S}^*(\boldsymbol{\alpha}) \equiv \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} - \lambda^* \,\mathrm{Var}\{\Psi_{\boldsymbol{\alpha}}\}; \tag{6.137}$$

and then to solve for the optimal allocation as follows:

$$\boldsymbol{\alpha}^* \equiv \underset{\boldsymbol{\alpha}\in\mathcal{C}}{\operatorname{argmax}}\, \mathcal{S}^*(\boldsymbol{\alpha}). \tag{6.138}$$

This is a quadratic function of the allocation, and thus an easier problem to solve than the two-step optimization (6.129)-(6.130).

Nevertheless, the definition of the pseudo-index of satisfaction (6.137) is incorrect, because it depends on the market through λ^*. In other words, the same investor displays different risk aversion coefficients λ^* when facing different markets. Therefore the pseudo-index of satisfaction (6.137) does not represent a description of the investor's preferences. Using it as if λ^* did not depend on the market might lead to nonsensical results.

Consider the previous example, where instead of the market (6.132) we have:

$$P_{T+\tau}^{(1)} \equiv 1, \quad P_{T+\tau}^{(2)} = \begin{cases} 1 \text{ (probability} = 50\%) \\ 3 \text{ (probability} = 50\%). \end{cases} \tag{6.139}$$

Then

$$\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\} = 1 + \alpha, \quad \mathrm{Var}\left\{\Psi_{\boldsymbol{\alpha}}\right\} = \alpha^2. \tag{6.140}$$

From the first-order condition in the Lagrange formulation (6.129) we obtain the mean-variance curve:

$$\alpha\left(\lambda\right) = \frac{1}{2\lambda}. \tag{6.141}$$

Since the optimal allocation is a full investment in the risky asset $\alpha^* \equiv 1$, in this market we obtain:

$$\lambda^* \equiv \frac{1}{2}. \tag{6.142}$$

Using in the mean-variance curve (6.141) the value (6.136) obtained in the previous market would give rise to a nonsensical positive allocation in the risk-free asset.

6.5.3 Quadratic programming and dual formulation

We recall that since all the indices of satisfaction $\mathcal{S}$ discussed in Chapter 5 are consistent with weak stochastic dominance, the mean-variance approach aims at maximizing the expected value of the investor's objective for a given level of variance. Therefore the mean-variance efficient allocations are the non-empty solutions of (6.68), which we report here:

$$\boldsymbol{\alpha}\left(v\right) \equiv \underset{\substack{\boldsymbol{\alpha}\in\mathcal{C} \\ \mathrm{Var}\left\{\Psi_{\boldsymbol{\alpha}}\right\}=v}}{\operatorname{argmax}} \mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\}, \tag{6.143}$$

where $v \geq 0$. Notice that the variance constraint appears as an equality.

Consider the plane of coordinates $v \equiv \mathrm{Var}\left\{\Psi_{\boldsymbol{\alpha}}\right\}$ and $e \equiv \mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\}$. If the upper limit of the feasible set determined by the constraints increases as we shift to the right on the horizontal axis, then (6.143) is equivalent to a problem with an inequality for the variance:

$$\boldsymbol{\alpha}\left(v\right) \equiv \underset{\substack{\boldsymbol{\alpha}\in\mathcal{C} \\ \mathrm{Var}\left\{\Psi_{\boldsymbol{\alpha}}\right\}\leq v}}{\operatorname{argmax}} \mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\}, \tag{6.144}$$

where $v \geq 0$.

This is not the case in the example in Figure 6.17: the allocations on the thick line in the north-east region would not be captured by (6.144), although they are efficient according to (6.143).

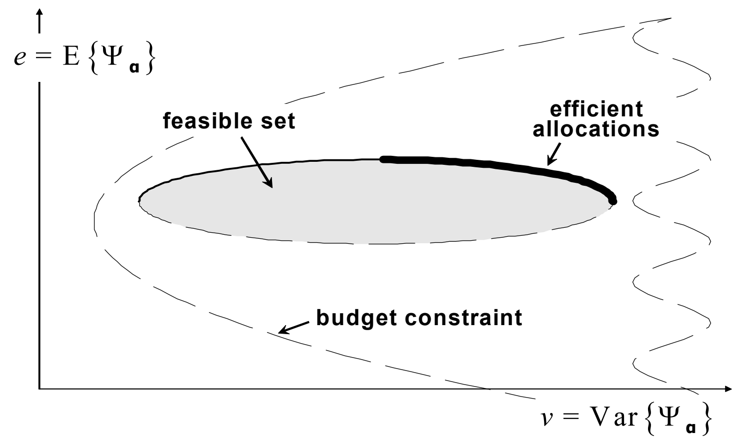

Fig. 6.17. MV efficient frontier as expected value maximization

On the other hand, we cannot rule out such allocations. For instance, in a prospect theoretical setting the investor becomes risk prone when facing losses, see the example in the shaded box on p. 269.

If the optimization with an inequality (6.144) is equivalent to the original problem (6.143) and if the investment constraints $\mathcal{C}$ are at most quadratic in the allocation, the optimization with an inequality is a quadratically constrained quadratic programming problem, see (6.57):

$$\boldsymbol{\alpha}(v) \equiv \underset{\substack{\boldsymbol{\alpha}\in\mathcal{C} \\ \boldsymbol{\alpha}' \operatorname{Cov}\{\mathbf{M}\}\boldsymbol{\alpha}\leq v}}{\operatorname{argmax}} \boldsymbol{\alpha}' \operatorname{E}\{\mathbf{M}\}, \tag{6.145}$$

where $v \geq 0$. Therefore this problem can be solved numerically.

At times the inequality-based mean-variance problem (6.144) is presented in its *dual formulation* as the non-empty set of the solutions to the following problem:

$$\boldsymbol{\alpha}(e) \equiv \underset{\substack{\boldsymbol{\alpha}\in\mathcal{C} \\ \operatorname{E}\{\Psi_{\boldsymbol{\alpha}}\}\geq e}}{\operatorname{argmin}} \operatorname{Var}\{\Psi_{\boldsymbol{\alpha}}\}, \tag{6.146}$$

where $e \in (-\infty, +\infty)$.

Under regularity conditions for the constraints $\mathcal{C}$ the dual formulation (6.146) is equivalent to (6.144), which in turn is equivalent to the original problem (6.143). The equivalence of these formulations must be checked on a case-by-case basis. For instance, the three formulations are equivalent when the constraints are affine, see (6.93), or for the standard no-short-sale constraint that appears in (6.80).

6.5.4 MV on returns: estimation versus optimization

In Section 6.3.4 we discussed how under the hypothesis (6.82) that the investor's objective is final wealth and the assumption (6.83) that the initial investment is not null, the general mean-variance formulation (6.74) is equivalent to the formulation (6.88) in terms of linear returns $\mathbf{L}$ and portfolio weights $\mathbf{w}$, defined in (6.81) and (6.86) respectively. We report here this formulation, emphasizing the realization time and the investment horizon in the notation for the linear returns:

$$\mathbf{w}(v) = \underset{\substack{\mathbf{w}\in\mathcal{C} \\ \mathbf{w}' \operatorname{Cov}\{\mathbf{L}_{T+\tau,\tau}\}\mathbf{w}=v}}{\operatorname{argmax}} \mathbf{w}' \operatorname{E}\{\mathbf{L}_{T+\tau,\tau}\}, \tag{6.147}$$

where $v \geq 0$.

In the process of computing the necessary inputs, namely the expected values $\operatorname{E}\{\mathbf{L}_{T+\tau,\tau}\}$ and the covariance matrix $\operatorname{Cov}\{\mathbf{L}_{T+\tau,\tau}\}$, there exists a clear distinction between estimation, which is performed on the market invariants $\mathbf{X}_{t,\widetilde{\tau}}$, and optimization, which acts on *functions* of the *projected* invariants $\mathbf{L}_{T+\tau,\tau}$, see the steps 1-6 on p. 324.

Now let us make two further assumptions. Assume that the market consists of equity-like securities as in Section 3.1.1, in which case the linear returns are market invariants:

$$\mathbf{X}_{t,\widetilde{\tau}} \equiv \mathbf{L}_{t,\widetilde{\tau}}. \tag{6.148}$$

Furthermore, assume that the investment horizon and the estimation interval coincide:

$$\tau \equiv \widetilde{\tau}. \tag{6.149}$$

Under the above combined assumptions it is possible to bypass many of the steps that lead to the inputs $\operatorname{E}\{\mathbf{L}_{T+\tau,\tau}\}$ and $\operatorname{Cov}\{\mathbf{L}_{T+\tau,\tau}\}$. Indeed, instead of estimating the whole distribution of the invariants $\mathbf{L}_{t,\widetilde{\tau}}$ as in Step 2 on p. 324, we estimate directly only its expected value $\operatorname{E}\{\mathbf{L}_{t,\widetilde{\tau}}\}$ and its covariance matrix $\operatorname{Cov}\{\mathbf{L}_{t,\widetilde{\tau}}\}$. Since by assumption the investment horizon is the estimation interval *and* since $\mathbf{L}_{t,\widetilde{\tau}}$ are invariants, the following holds:

$$\operatorname{E}\{\mathbf{L}_{T+\tau,\tau}\} = \operatorname{E}\{\mathbf{L}_{t,\widetilde{\tau}}\}, \quad \operatorname{Cov}\{\mathbf{L}_{T+\tau,\tau}\} = \operatorname{Cov}\{\mathbf{L}_{t,\widetilde{\tau}}\}. \tag{6.150}$$

Therefore we can skip Step 3, Step 4, Step 5 and Step 6 and plug (6.150) directly in the mean-variance problem (6.147).

We stress that the above shortcut is not viable in general. For instance, in the fixed-income market the linear returns are not market invariants. Instead, the market invariants are the changes in yield to maturity, see Section 3.1.2. Therefore in order to perform the mean-variance analysis in the fixed-income market in terms of relative weights and linear returns we need to go through all the steps 1-6 on p. 324. Nonetheless, one is dangerously tempted to estimate the returns as if they were invariants and proceed with the shortcut (6.150).

6.5.5 MV on returns: investment at different horizons

Another misunderstanding regarding the mean-variance framework occurs when the investment horizon τ is shifted farther in the future and the mean-variance optimization is formulated in terms of returns, see also Meucci (2001).

As in (6.147) we make the assumptions that the investor's objective is final wealth and that the initial investment is not zero, in such a way that the general mean-variance formulation (6.74) is equivalent to the formulation in term of linear returns and portfolio weights.

As in (6.148) we consider the case where the market consists of equity-like securities, in which case the linear returns are market invariants.

Nevertheless, unlike (6.149), we consider an investment horizon that is different, typically longer, than the estimation interval:

$$\tau > \widetilde{\tau}. \tag{6.151}$$

In this case the shortcut (6.150) does not apply. Instead, we need to project the distribution of the invariants to the investment horizon and then compute the quantities of interest $\mathrm{E}\{\mathbf{L}_{T+\tau,\tau}\}$ and $\mathrm{Cov}\{\mathbf{L}_{T+\tau,\tau}\}$ as described in the steps 1-6 on p. 324. As we see below, only when the market is not too volatile and both the investment horizon and the estimation interval are short is the shortcut (6.150) approximately correct, see also Meucci (2004).

Since we are dealing with equity-like securities, the projection of the invariants into the moments of the linear returns takes a simpler form than in the more general case discussed in Section 6.3.4. This is the same argument that leads to (6.78) and (6.79) in the mean-variance formulation in terms of prices. Therefore it applies also to the fixed-income market. Here we present this argument explicitly in the case of equity-like securities.

We recall from (3.11) that for a generic security or portfolio that is worth P_t at time t, the τ-horizon compounded return at time t is defined as follows:

$$C_{t,\tau} \equiv \ln\left(\frac{P_t}{P_{t-\tau}}\right). \tag{6.152}$$

Therefore the linear returns (6.81) are the following function of the compounded returns:

$$1 + L_{t,\tau} \equiv e^{C_{t,\tau}}. \tag{6.153}$$

From the above equality we obtain the following relation for the expected value of the linear returns:

$$\begin{aligned} \mathrm{E}\left\{1 + L^{(n)}_{T+\tau,\tau}\right\} &= \mathrm{E}\left\{e^{C^{(n)}_{T+\tau,\tau}}\right\} \\ &= \phi_{\mathbf{C}_{T+\tau,\tau}}\left(-i\boldsymbol{\delta}^{(n)}\right). \end{aligned} \tag{6.154}$$

In this expression $\phi_{\mathbf{C}_{T+\tau,\tau}}$ is the joint characteristic function of the compounded returns relative to the investment horizon and $\boldsymbol{\delta}^{(n)}$ is the n-th element canonical basis $(A.15)$, i.e. it is a vector of zeros, except for the n-th entry, which is one. Similarly, from (6.153) we obtain:

$$\mathrm{E}\left\{\left(1+L_{T+\tau,\tau}^{(m)}\right)\left(1+L_{T+\tau,\tau}^{(n)}\right)\right\} = \mathrm{E}\left\{e^{C_{T+\tau,\tau}^{(m)}+C_{T+\tau,\tau}^{(n)}}\right\} \tag{6.155}$$
$$= \phi_{\mathbf{C}_{T+\tau,\tau}}\left(-i\left(\boldsymbol{\delta}^{(m)}+\boldsymbol{\delta}^{(n)}\right)\right).$$

From these expressions in turn we immediately obtain the desired quantities:

$$\mathrm{E}\left\{L_{T+\tau,\tau}^{(n)}\right\} = \phi_{\mathbf{C}_{T+\tau,\tau}}\left(-i\boldsymbol{\delta}^{(n)}\right)-1 \tag{6.156}$$

and

$$\mathrm{Cov}\left\{L_{T+\tau,\tau}^{(m)}, L_{T+\tau,\tau}^{(n)}\right\} = \mathrm{Cov}\left\{1+L_{T+\tau,\tau}^{(m)}, 1+L_{T+\tau,\tau}^{(n)}\right\} \tag{6.157}$$
$$= \phi_{\mathbf{C}_{T+\tau,\tau}}\left(-i\left(\boldsymbol{\delta}^{(m)}+\boldsymbol{\delta}^{(n)}\right)\right)$$
$$-\phi_{\mathbf{C}_{T+\tau,\tau}}\left(-i\boldsymbol{\delta}^{(m)}\right)\phi_{\mathbf{C}_{T+\tau,\tau}}\left(-i\boldsymbol{\delta}^{(n)}\right).$$

Therefore, in order to compute the inputs of the mean-variance optimization (6.156) and (6.157) we need to derive the expression of the characteristic function $\phi_{\mathbf{C}_{T+\tau,\tau}}$ from the distribution of the market invariants (6.148). In order to do this, we notice that if the linear returns $\mathbf{L}_{t,\widetilde{\tau}}$ are market invariants, so are the compounded returns $\mathbf{C}_{t,\widetilde{\tau}}$. For the compounded returns the simple projection formula (3.64) holds, which in this context reads:

$$\phi_{\mathbf{C}_{T+\tau,\tau}} = \left(\phi_{\mathbf{C}_{t,\widetilde{\tau}}}\right)^{\frac{\tau}{\widetilde{\tau}}}. \tag{6.158}$$

Notice that this formula does not hold for the linear returns, whose projection formula relies on the much more complex expression (3.78).

Substituting (6.158) into (6.156) and (6.157) we obtain the desired inputs of the mean-variance problem directly in terms of the distribution of the market invariants:

$$\mathrm{E}\left\{L_{T+\tau,\tau}^{(n)}\right\} = \left[\phi_{\mathbf{C}_{t,\widetilde{\tau}}}\left(-i\boldsymbol{\delta}^{(n)}\right)\right]^{\frac{\tau}{\widetilde{\tau}}}-1 \tag{6.159}$$

and

$$\mathrm{Cov}\left\{L_{T+\tau,\tau}^{(m)}, L_{T+\tau,\tau}^{(n)}\right\} = \left[\phi_{\mathbf{C}_{t,\widetilde{\tau}}}\left(-i\left(\boldsymbol{\delta}^{(m)}+\boldsymbol{\delta}^{(n)}\right)\right)\right]^{\frac{\tau}{\widetilde{\tau}}} \tag{6.160}$$
$$-\left[\phi_{\mathbf{C}_{t,\widetilde{\tau}}}\left(-i\boldsymbol{\delta}^{(m)}\right)\right]^{\frac{\tau}{\widetilde{\tau}}}\left[\phi_{\mathbf{C}_{t,\widetilde{\tau}}}\left(-i\boldsymbol{\delta}^{(n)}\right)\right]^{\frac{\tau}{\widetilde{\tau}}}.$$

For instance, assuming as in Black and Scholes (1973) that the compounded returns are normally distributed, from (2.157) we obtain their characteristic function:

$$\phi_{\mathbf{C}_{t,\widetilde{\tau}}}(\boldsymbol{\omega}) = e^{i\boldsymbol{\omega}'\boldsymbol{\mu}-\frac{1}{2}\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}}. \tag{6.161}$$

Therefore from (6.159) the expected values read:

$$\mathrm{E}\left\{L_{T+\tau,\tau}^{(n)}\right\} = e^{\frac{\tau}{\tilde{\tau}}\left(\mu_n + \frac{1}{2}\Sigma_{nn}\right)} - 1; \tag{6.162}$$

and from (6.160) the covariances read:

$$\mathrm{Cov}\left\{L_{T+\tau,\tau}^{(m)}, L_{T+\tau,\tau}^{(n)}\right\} = e^{\frac{\tau}{\tilde{\tau}}\left(\mu_m + \mu_n + \frac{1}{2}\Sigma_{mm} + \frac{1}{2}\Sigma_{nn}\right)}\left(e^{\frac{\tau}{\tilde{\tau}}\Sigma_{mn}} - 1\right). \tag{6.163}$$

The reader is invited to consider the limit of the expressions (6.162) and (6.163) when the market volatility is low and the investment horizon τ is short.

Instead of using the correct formulas (6.159) and (6.160) in the mean-variance optimization (6.147), some practitioners replace the linear returns with compounded returns. In other words, they *define* the mean-variance efficient frontier as follows:

$$\widetilde{\mathbf{w}}(v) \equiv \underset{\substack{\mathbf{w}\in\mathcal{C} \\ \mathbf{w}'\,\mathrm{Cov}\{\mathbf{C}_{T+\tau,\tau}\}\mathbf{w}=v}}{\operatorname{argmax}} \mathbf{w}'\,\mathrm{E}\left\{\mathbf{C}_{T+\tau,\tau}\right\}, \tag{6.164}$$

where $v \geq 0$. In this formulation, the "square-root rule" (3.75) and (3.76), which is a consequence of (6.158), applies:

$$\mathrm{E}\left\{\mathbf{C}_{T+\tau,\tau}\right\} = \frac{\tau}{\widetilde{\tau}}\,\mathrm{E}\left\{\mathbf{C}_{t,\widetilde{\tau}}\right\}, \quad \mathrm{Cov}\left\{\mathbf{C}_{T+\tau,\tau}\right\} = \frac{\tau}{\widetilde{\tau}}\,\mathrm{Cov}\left\{\mathbf{C}_{t,\widetilde{\tau}}\right\}. \tag{6.165}$$

Therefore, it suffices to estimate the expected values and the covariance matrix of the compounded return for a given estimation interval $\widetilde{\tau}$, and use the results for any investment horizon τ.

Nevertheless, the definition (6.164) of the mean-variance problem is incorrect.

In the first place, unlike the formulation in terms of linear returns (6.147), this formulation is not equivalent to the general mean-variance problem (6.74), because the identity (6.87) does not hold for the compounded returns:

$$C_{T+\tau,\tau}^{\Psi_\alpha} \neq \mathbf{w}'\mathbf{C}_{T+\tau,\tau}. \tag{6.166}$$

More in general, the quantity $\mathbf{w}'\mathbf{C}_{T+\tau,\tau}$ does not represent any feature of the investor's portfolio, not only it does not represent its compounded return. Therefore also the quantities $\mathbf{w}'\,\mathrm{E}\{\mathbf{C}\}$ and $\mathbf{w}'\,\mathrm{Cov}\left\{\mathbf{C}_{T+\tau,\tau}\right\}\mathbf{w}$ that appear in (6.164) are not related to the investor's portfolio.

Secondly, from the square-root rule (6.165) it follows that the mean-variance efficient allocations (6.164) do not depend on the investment horizon. This is incorrect and counterintuitive.

Notice that in the case of a short investment horizon τ and a not-too-volatile market, a first-order Taylor expansion shows that the linear and the compounded returns are approximately the same:

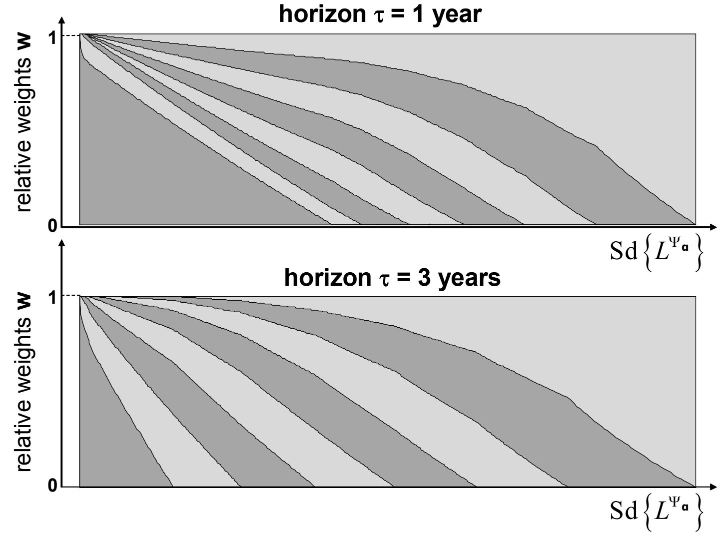

Fig. 6.18. MV efficient allocations at different investment horizons

$$L_{T+\tau,\tau} \equiv \frac{P_{T+\tau}}{P_T} - 1 \approx \ln\left(\frac{P_{T+\tau}}{P_T}\right) \equiv C_{T+\tau,\tau}. \tag{6.167}$$

In this case the correct mean-variance efficient allocations obtained from the formulation in terms of the linear returns (6.147) are approximately equal to the mean-variance efficient allocations obtained from the formulation in terms of the compounded returns (6.164), and therefore they are approximately independent of the investment horizon τ:

$$\begin{aligned}
\mathbf{w}(v) \equiv & \underset{\substack{\mathbf{w}\in\mathcal{C} \\ \mathbf{w}'\operatorname{Cov}\{\mathbf{L}_{T+\tau,\tau}\}\mathbf{w}=v}}{\operatorname{argmax}} \mathbf{w}'\operatorname{E}\{\mathbf{L}_{T+\tau,\tau}\} \\
\approx & \underset{\substack{\mathbf{w}\in\mathcal{C} \\ \mathbf{w}'\operatorname{Cov}\{\mathbf{C}_{T+\tau,\tau}\}\mathbf{w}=v}}{\operatorname{argmax}} \mathbf{w}'\operatorname{E}\{\mathbf{C}_{T+\tau,\tau}\} \\
= & \underset{\substack{\mathbf{w}\in\mathcal{C} \\ \mathbf{w}'\operatorname{Cov}\left\{\mathbf{C}_{T,\widetilde{\tau}}\right\}\mathbf{w}=\frac{\widetilde{\tau}}{\tau}v}}{\operatorname{argmax}} \frac{\tau}{\widetilde{\tau}}\mathbf{w}'\operatorname{E}\{\mathbf{C}_{t,\widetilde{\tau}}\} \\
= & \widetilde{\mathbf{w}}(s),
\end{aligned} \tag{6.168}$$

where $s \equiv v\widetilde{\tau}/\tau \geq 0$. Indeed, in the limit case of a dynamic setting, where the investor can rebalance continuously his portfolio and thus the investment horizon tends to zero, the formulation in terms of compounded returns becomes correct, and an equality holds in (6.166), see Merton (1992).

Nevertheless, for longer investment horizons and more volatile markets the first-order Taylor approximation (6.167) is not accurate. Indeed, the efficient frontiers relative to different investment horizons are different.

We see this Figure 6.18, where we plot the efficient combination of eight securities at a horizon of one and three years respectively under the normal assumption (6.161). Refer to `symmys.com` for the details on this market of securities and on the computations that generated the plots.

6.6 Total-return versus benchmark allocation

In this section we present an application of the analytical solutions of the mean-variance problem discussed in Section 6.4. We analyze two standard allocation strategies in the mutual fund industry: total-return allocation and benchmark allocation. As it turns out, benchmark allocation is the implicit strategy of the generic investor.

In the total-return strategy the investor's objective is final wealth at the investment horizon, see (5.3):

$$\Psi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}' \mathbf{P}_{T+\tau}. \tag{6.169}$$

In the benchmark strategy the investor's objective is to overperform a benchmark whose allocation is $\widetilde{\boldsymbol{\beta}}$. In this case, the investor's objective is the *overperformance*, see (5.4):

$$\Phi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}' \mathbf{P}_{T+\tau} - \gamma \widetilde{\boldsymbol{\beta}}' \mathbf{P}_{T+\tau}, \tag{6.170}$$

where the normalization scalar is meant to make the comparison between the portfolio and the benchmark fair:

$$\gamma \equiv \frac{\boldsymbol{\alpha}' \mathbf{p}_T}{\widetilde{\boldsymbol{\beta}}' \mathbf{p}_T}. \tag{6.171}$$

In both the total-return and the benchmark strategies the investor is bound by the same budget constraint:

$$\mathcal{C}: \ \boldsymbol{\alpha}' \mathbf{p}_T = w > 0. \tag{6.172}$$

In order to cast the total-return allocation problem in the mean-variance framework, we assume as in (6.67) that the investor's satisfaction only depends on the first two moments of his objective, namely final wealth:

$$\mathcal{S}(\boldsymbol{\alpha}) \approx \widetilde{\mathcal{H}}\left(\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\}, \mathrm{Var}\left\{\Psi_{\boldsymbol{\alpha}}\right\}\right), \tag{6.173}$$

where $\widetilde{\mathcal{H}}$ is a suitable bivariate function. Given the constraint (6.172), the mean-variance efficient frontier solves an affine constraint problem of the form (6.96), which in this context reads:

$$\widetilde{\boldsymbol{\alpha}}(v) = \underset{\substack{\boldsymbol{\alpha}'\mathbf{p}_T = w \\ \operatorname{Var}\{\Psi_{\boldsymbol{\alpha}}\} = v}}{\operatorname{argmax}} \operatorname{E}\{\Psi_{\boldsymbol{\alpha}}\}, \tag{6.174}$$

where $v \geq 0$.

The non-empty solutions of this optimization, namely the total-return efficient frontier, follow from (6.97). They represent a straight semi-line parametrized by the expected value of final wealth $e \equiv \operatorname{E}\{\Psi_{\widetilde{\boldsymbol{\alpha}}}\}$, which we report here:

$$\widetilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha}_{MV} + [e - \operatorname{E}\{\Psi_{\boldsymbol{\alpha}_{MV}}\}] \frac{\boldsymbol{\alpha}_{SR} - \boldsymbol{\alpha}_{MV}}{\operatorname{E}\{\Psi_{\boldsymbol{\alpha}_{SR}}\} - \operatorname{E}\{\Psi_{\boldsymbol{\alpha}_{MV}}\}}, \tag{6.175}$$

where $e \in [\operatorname{E}\{\Psi_{\boldsymbol{\alpha}_M}\}, +\infty)$. The global minimum variance portfolio and the maximum Sharpe ratio portfolio in this expression follow from (6.99) and (6.100) respectively and read in this context as follows:

$$\boldsymbol{\alpha}_{MV} \equiv \frac{w \operatorname{Cov}\{\mathbf{P}_{T+\tau}\}^{-1} \mathbf{p}_T}{\mathbf{p}_T' \operatorname{Cov}\{\mathbf{P}_{T+\tau}\}^{-1} \mathbf{p}_T} \tag{6.176}$$

$$\boldsymbol{\alpha}_{SR} \equiv \frac{w \operatorname{Cov}\{\mathbf{P}_{T+\tau}\}^{-1} \operatorname{E}\{\mathbf{P}_{T+\tau}\}}{\mathbf{p}_T' \operatorname{Cov}\{\mathbf{P}_{T+\tau}\}^{-1} \operatorname{E}\{\mathbf{P}_{T+\tau}\}}. \tag{6.177}$$

In order to cast the benchmark allocation problem in the mean-variance framework, we first introduce some jargon used by practitioners. The expected value of the investor's objective (6.170) is called *expected overperformance*, which we denote as follows:

$$\operatorname{EOP}(\boldsymbol{\alpha}) \equiv \operatorname{E}\{\Phi_{\boldsymbol{\alpha}}\}. \tag{6.178}$$

The standard deviation of the investor's objective is called the *tracking error*[1], which we denote as follows:

$$\operatorname{TE}(\boldsymbol{\alpha}) \equiv \operatorname{Sd}\{\Phi_{\boldsymbol{\alpha}}\}. \tag{6.179}$$

The Sharpe ratio, i.e. the ratio of the above two parameters, is called the *information ratio*, which we denote as follows:

$$\operatorname{IR}(\boldsymbol{\alpha}) \equiv \frac{\operatorname{EOP}(\boldsymbol{\alpha})}{\operatorname{TE}(\boldsymbol{\alpha})}. \tag{6.180}$$

[1] Some authors define the tracking error differently: Roll (1992) defines it as the overperformance:

$$\operatorname{TE}(\boldsymbol{\alpha}) \equiv \Phi_{\boldsymbol{\alpha}};$$

Leibowitz, Bader, and Kogelman (1996) define it as follows:

$$\operatorname{TE}\{\boldsymbol{\alpha}\} \equiv \sqrt{\operatorname{E}\{\Phi_{\boldsymbol{\alpha}}^2\}}.$$

As in (6.67), we assume that the satisfaction of an investor whose purpose is to overperform a benchmark only depends on the first two moments of his objective:

$$\mathcal{S}(\boldsymbol{\alpha}) \approx \widetilde{\mathcal{K}}\left(\mathrm{EOP}(\boldsymbol{\alpha}), \mathrm{TE}^2(\boldsymbol{\alpha})\right), \tag{6.181}$$

where $\widetilde{\mathcal{K}}$ is a suitable bivariate function. Therefore the mean-variance efficient frontier of the benchmark strategy solves an affine constraint problem of the form (6.96), which in the newly introduced notation reads:

$$\widehat{\boldsymbol{\alpha}}(u) = \underset{\substack{\boldsymbol{\alpha}'\mathbf{p}_T = w \\ \mathrm{TE}^2(\boldsymbol{\alpha}) = u}}{\operatorname{argmax}} \ \mathrm{EOP}(\boldsymbol{\alpha}), \tag{6.182}$$

where $u \geq 0$.

To solve this problem we could cast the benchmark-relative objective in the form $\Phi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}'\mathbf{M}$ for the market vector defined in (5.11) and (5.13) and then write the solution in the form (6.97). Nevertheless, we gain more insight into the differences and similarities between total-return allocation and benchmark allocation if we re-formulate the benchmark problem in terms of relative bets, which represent the difference between the allocation chosen by the investor and the allocation of the benchmark.

First of all we define the normalized benchmark allocation as follows:

$$\boldsymbol{\beta} \equiv \frac{w}{\mathbf{p}_T'\widetilde{\boldsymbol{\beta}}}\widetilde{\boldsymbol{\beta}}. \tag{6.183}$$

It is immediate to check that the normalized benchmark is a rescaled version of the original benchmark which has the same value as the investor's portfolio at the time the investment is made. The relative bets are defined as the vector $\boldsymbol{\rho}$ such that:

$$\boldsymbol{\alpha} \equiv \boldsymbol{\beta} + \boldsymbol{\rho}. \tag{6.184}$$

Since the benchmark allocation $\boldsymbol{\beta}$ is fixed, it is equivalent to determine and express the efficient frontier in terms of the allocations $\boldsymbol{\alpha}$ or in terms of the relative bets $\boldsymbol{\rho}$.

In terms of the relative bets, it is easy to check that the benchmark-relative objective (6.170) takes the form of a total-return objective:

$$\Phi_{\boldsymbol{\alpha}} = (\boldsymbol{\alpha} - \boldsymbol{\beta})'\,\mathbf{P}_{T+\tau} = \boldsymbol{\rho}'\mathbf{P}_{T+\tau} \equiv \Psi_{\boldsymbol{\rho}}, \tag{6.185}$$

compare with (6.169). Furthermore, the budget constraint (6.172) simplifies to a linear constraint:

$$\mathcal{C}: \ \boldsymbol{\rho}'\mathbf{p}_T = (\boldsymbol{\alpha} - \boldsymbol{\beta})'\,\mathbf{p}_T = 0. \tag{6.186}$$

Therefore the efficient frontier of the benchmark strategy (6.182) can be written in terms of the relative bets that solve the following problem:

$$\widehat{\boldsymbol{\rho}}(u) = \underset{\substack{\boldsymbol{\rho}'\mathbf{p}_T = 0 \\ \mathrm{Var}\{\Psi_{\boldsymbol{\rho}}\} = u}}{\operatorname{argmax}} \ \mathrm{E}\{\Psi_{\boldsymbol{\rho}}\}, \tag{6.187}$$

where $u \geq 0$. This linear constraint problem is of the from (6.107). The non-empty solutions to this optimization, namely the benchmark-relative efficient frontier, follow from (6.109) and (6.111). They represent a straight semi-line parametrized by the expected value of the overperformance $p \equiv \mathrm{E}\left\{\Psi_{\widehat{\boldsymbol{\rho}}}\right\}$:

$$\widehat{\boldsymbol{\rho}} = p\frac{\boldsymbol{\alpha}_{SR} - \boldsymbol{\alpha}_{MV}}{\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}_{SR}}\right\} - \mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}_{MV}}\right\}}, \tag{6.188}$$

where $p \geq 0$.

From the definition of the relative bets (6.184) and the definition of the overperformance (6.185) we recover the efficient frontier of the benchmark strategy. This is a straight line parameterized by the expected final wealth of the benchmark-relative efficient allocations:

$$e \equiv \mathrm{E}\left\{\Psi_{\widehat{\boldsymbol{\alpha}}}\right\} \in \left[\mathrm{E}\left\{\Psi_{\boldsymbol{\beta}}\right\}, +\infty\right), \tag{6.189}$$

and reads explicitly:

$$\widehat{\boldsymbol{\alpha}} = \boldsymbol{\beta} + \left[e - \mathrm{E}\left\{\Psi_{\boldsymbol{\beta}}\right\}\right] \frac{\boldsymbol{\alpha}_{SR} - \boldsymbol{\alpha}_{MV}}{\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}_{SR}}\right\} - \mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}_{MV}}\right\}}. \tag{6.190}$$

A comparison of this expression with (6.175) shows that the total-return allocations $\widetilde{\boldsymbol{\alpha}}$ can be interpreted as benchmark-relative allocations $\widehat{\boldsymbol{\alpha}}$, where the benchmark is represented by the global minimum-variance portfolio.

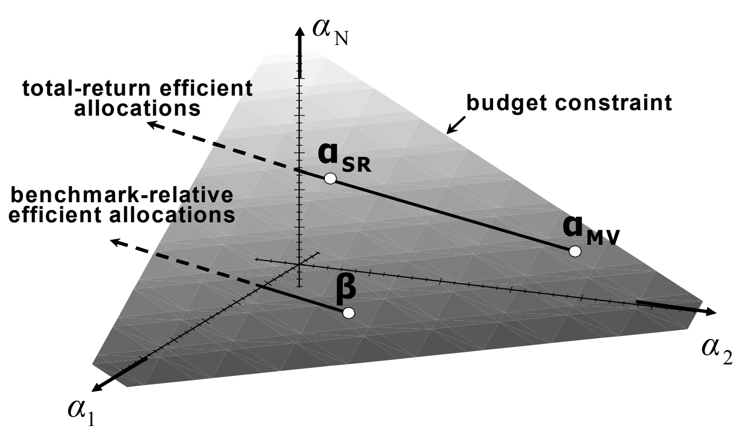

Fig. 6.19. Total-return vs. benchmark-relative MV efficient allocations

Geometrically, we can interpret these results as in Figure 6.19. In the N-dimensional space of allocations, the mean-variance efficient frontiers of both the total-return strategy and the benchmark-relative strategy are straight semi-lines that lie in the $(N-1)$-dimensional hyperplane determined by the budget constraint.

The direction of departure of the efficient straight semi-line from its starting point is determined by a two-fund separation principle. This direction of departure is the same for both the total-return strategy and the benchmark-relative strategy, no matter the specific composition of the benchmark.

On the other hand, the starting point of the efficient straight line is the investor's benchmark. In particular, in the total-return case where the investor's objective is final wealth, the starting point of the efficient straight line is the global minimum-variance portfolio.

We can now analyze the satisfaction stemming from the efficient allocations, see Roll (1992). First we consider the satisfaction of the total-return investor (6.173), which depends on the first two moments of the distribution of final wealth:

$$\begin{aligned} v &\equiv \operatorname{Var}\{\Psi_{\boldsymbol{\alpha}}\} \qquad (6.191)\\ &= \boldsymbol{\alpha}' \operatorname{Cov}\{\mathbf{P}_{T+\tau}\}\, \boldsymbol{\alpha} \\ e &\equiv \operatorname{E}\{\Psi_{\boldsymbol{\alpha}}\} \qquad (6.192)\\ &= \boldsymbol{\alpha}' \operatorname{E}\{\mathbf{P}_{T+\tau}\}. \end{aligned}$$

Since both the total-return investor and the benchmark-relative investor share the same *affine* budget constraint (6.172), their feasible set is the same. Like in Figure 6.10, the feasible set plots as the internal portion of a parabola, see Figure 6.20.

The total-return mean-variance efficient allocations (6.175) generate the upper branch of this parabola. From (6.102) the equation of this parabola reads:

$$\widetilde{\boldsymbol{\alpha}}: \quad v = \frac{A}{D}e^2 - \frac{2wB}{D}e + \frac{w^2 C}{D}, \qquad (6.193)$$

where (A, B, C, D) are the four scalars (6.103) that do not depend on the allocation and which in this context read:

$$\begin{aligned} A &\equiv \mathbf{p}_T' \operatorname{Cov}\{\mathbf{P}_{T+\tau}\}^{-1} \mathbf{p}_T \\ B &\equiv \mathbf{p}_T' \operatorname{Cov}\{\mathbf{P}_{T+\tau}\}^{-1} \operatorname{E}\{\mathbf{P}_{T+\tau}\} \\ C &\equiv \operatorname{E}\{\mathbf{P}_{T+\tau}\}' \operatorname{Cov}\{\mathbf{P}_{T+\tau}\}^{-1} \operatorname{E}\{\mathbf{P}_{T+\tau}\} \\ D &\equiv AC - B^2. \end{aligned} \qquad (6.194)$$

The benchmark-relative efficient allocations (6.190) are sub-optimal in these coordinates and thus do not lie on the upper branch of the parabola. As we prove in Appendix www.6.5, the benchmark-relative optimal allocations give rise to the portion of a parabola above the coordinates of the benchmark, see Figure 6.20. This parabola represents a right translation of the mean-variance efficient parabola (6.193) generated by the total-return efficient allocations. Indeed, the equation of this parabola reads:

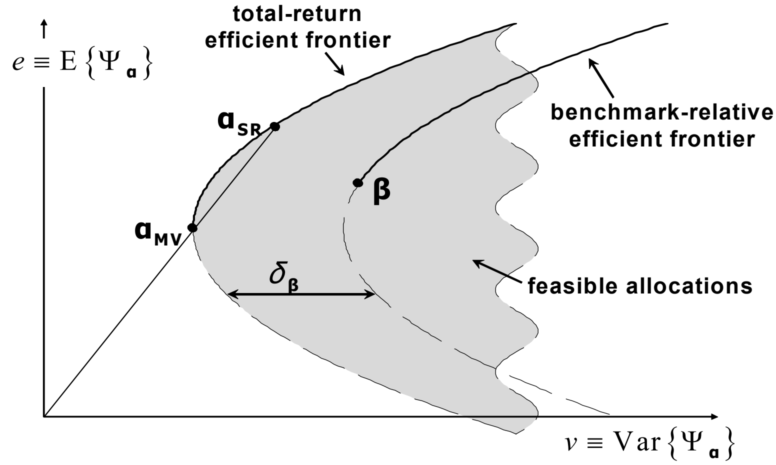

Fig. 6.20. Risk/reward profile of efficient allocations: total-return coordinates

$$\widehat{\boldsymbol{\alpha}}: \quad v = \frac{A}{D}e^2 - \frac{2wB}{D}e + \frac{w^2 C}{D} + \delta_{\boldsymbol{\beta}}. \tag{6.195}$$

In this expression $\delta_{\boldsymbol{\beta}}$ is a benchmark-dependent, non-negative scalar:

$$\delta_{\boldsymbol{\beta}} \equiv \mathrm{Var}\{\Psi_{\boldsymbol{\beta}}\} - \frac{A}{D}\,\mathrm{E}\{\Psi_{\boldsymbol{\beta}}\}^2 + \frac{2wB}{D}\,\mathrm{E}\{\Psi_{\boldsymbol{\beta}}\} - \frac{w^2 C}{D} \geq 0, \tag{6.196}$$

As we show in Appendix www.6.5, the equality holds in (6.196) if and only if the benchmark is mean-variance efficient from a total-return point of view.

Now we discuss the satisfaction (6.181) of the investor who aims at outperforming a benchmark, which depends on the first two moments of the overperformance:

$$\begin{aligned} u &\equiv \mathrm{Var}\{\Phi_{\boldsymbol{\alpha}}\} \equiv \mathrm{TE}^2(\boldsymbol{\alpha}) \qquad (6.197)\\ &= (\boldsymbol{\alpha} - \boldsymbol{\beta})' \,\mathrm{Cov}\{\mathbf{P}_{T+\tau}\}(\boldsymbol{\alpha} - \boldsymbol{\beta}) \\ p &\equiv \mathrm{E}\{\Phi_{\boldsymbol{\alpha}}\} \equiv \mathrm{EOP}(\boldsymbol{\alpha}) \qquad (6.198)\\ &= (\boldsymbol{\alpha} - \boldsymbol{\beta})' \,\mathrm{E}\{\mathbf{P}_{T+\tau}\}. \end{aligned}$$

Since both the total-return investor and the benchmark-relative investor share the *linear* budget constraint (6.186), their feasible set is the same. From the discussion on p. 331, the feasible set plots as the internal region of a parabola through the origin, which represents an allocation that fully replicates the benchmark, see Figure 6.21.

The benchmark-relative efficient allocations (6.190) generate the upper branch of this parabola. From (6.112) the equation of the parabola reads:

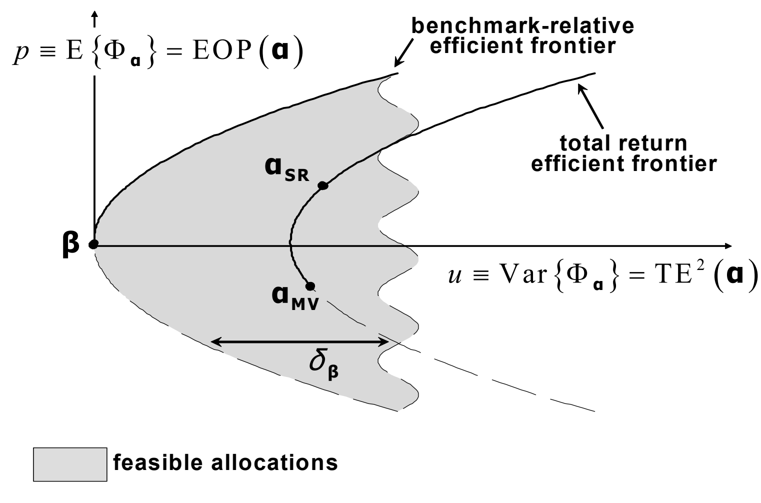

Fig. 6.21. Risk/reward profile of efficient allocations: benchmark-relative coordinates

$$\widehat{\boldsymbol{\alpha}}: \quad u = \frac{A}{D} p^2, \tag{6.199}$$

where A and D are defined in (6.194).

The total-return mean-variance efficient allocations (6.175) are sub-optimal in these coordinates, and thus they do not lie on the optimal parabola. In Appendix www.6.5 we prove that the total-return mean-variance efficient allocations give rise to the portion of a parabola above the coordinates of the global minimum-variance portfolio, see Figure 6.21. This parabola represents a right translation of the parabola (6.199) generated by the benchmark-relative efficient allocations. Indeed the equation of this parabola reads:

$$\widetilde{\boldsymbol{\alpha}}: \quad u = \frac{A}{D} p^2 + \delta_{\boldsymbol{\beta}}. \tag{6.200}$$

In this expression $\delta_{\boldsymbol{\beta}}$ is the same non-negative constant as (6.196), which is null if and only if the benchmark is mean-variance efficient from a total-return point of view.

Finally, it is interesting to look at the risk/reward profile of the total-return and benchmark-relative efficient allocations in the plane of the tracking error and expected overperformance plane, see Figure 6.22. Much like in (6.112), in these coordinates the boundary of the feasible set degenerates into two straight lines.

The total-return efficient allocations plot as a portion of the upper branch of a hyperbola within the feasible set. On the other hand, the benchmark-relative efficient allocations represent the straight line in the positive quadrant

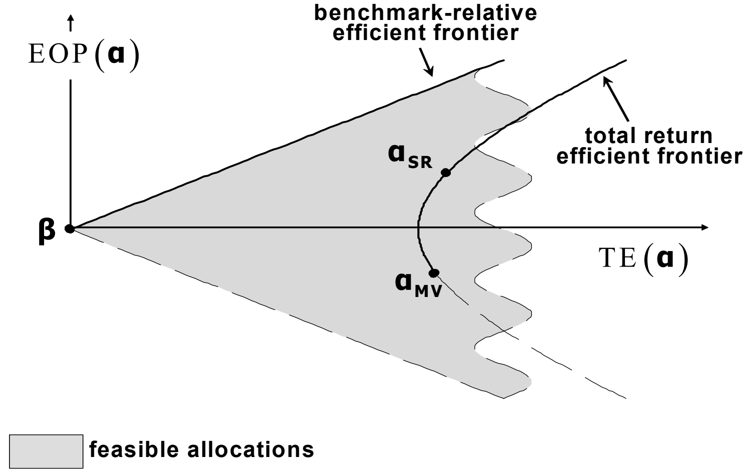

Fig. 6.22. Risk-reward profile of efficient allocations: expected overperformance and tracking error

that limits the feasible set. From (6.199) and the definitions (6.197)-(6.198), the equation of this line reads:

$$\mathrm{EOP}(\widehat{\boldsymbol{\alpha}}) = \sqrt{\frac{D}{A}}\,\mathrm{TE}(\widehat{\boldsymbol{\alpha}}) . \tag{6.201}$$

As we see in Figure 6.22, such allocations give rise to the highest possible information ratio, defined in (6.180): as the investor is willing to accept a larger tracking error, the attainable expected overperformance increases linearly.

Depending on their specific index of satisfaction, some investors will abandon the benchmark, aggressively pursuing a higher expected overperformance. On the other hand, other investors will closely track the benchmark, minimizing their relative risk: this is the case for *index funds*, whose aim is to replicate the performance of a benchmark at the minimum possible cost. Refer to Section 3.4.5 for a routine to implement portfolio replication.

6.7 Case study: allocation in stocks

In this section we revisit all the steps of a real allocation problem that lead to the optimal portfolio for a given investor. Unlike the leading example in Section 6.1, this case study cannot be solved analytically. Therefore, after collecting the necessary information on the investor and on his market, we simplify the problem according to the two-step mean variance recipe, as discussed in Section 6.3 and we compute the optimal portfolio numerically.

We stress that at this stage little importance is given to the yet very important issue of estimation risk. We discuss this issue in depth in the third part of the book.

6.7.1 Collecting information on the investor

The investor starts in general with a pre-existing allocation $\boldsymbol{\alpha}^{(0)}$. In this case we assume that the investor's initial wealth is a given amount of cash w, say ten thousand dollars.

The investor determines a market. We assume that he chooses a set of $N \equiv 8$ well-diversified stocks and that he plans to re-invest any dividends. We denote as $\mathbf{P}_t$ the prices at the generic time t of one share of the stocks.

The investor determines his investment horizon τ: in this example we set τ equal to one year.

The investor specifies his objective. In this example we assume that he focuses on final wealth:

$$\Psi_{\boldsymbol{\alpha}} \equiv \boldsymbol{\alpha}' \mathbf{P}_{T+\tau}, \tag{6.202}$$

where T denotes the current time.

We model the investor's satisfaction. We assume that the investor bases his decisions according to the certainty-equivalent of his expected utility as in (5.93), where his utility function is of the power type. Therefore the investor's satisfaction reads:

$$\mathcal{S}(\boldsymbol{\alpha}) \equiv \left(\gamma \operatorname{E} \left\{ \frac{\Psi_{\boldsymbol{\alpha}}^{\gamma}}{\gamma} \right\} \right)^{\frac{1}{\gamma}}. \tag{6.203}$$

Notice that the power utility function is defined only for positive values of the investor's objective. This is consistent with the fact that prices are positive and that the investor can only hold long positions in the stocks. In our example we set the specific value $\gamma \equiv -9$ for the risk aversion parameter of the power utility functions.

6.7.2 Collecting information on the market

In order to collect information on the market we turn to data providers and we retrieve the time-series of the stock prices, which are available, say, for the past five years.

We determine the market invariants. After performing the analysis of Section 3.1, we determine that the non-overlapping compounded returns of the stocks can be modeled as independent and identically distributed across time:

$$C_{t,\widetilde{\tau}}^{(n)} \equiv \ln \left(\frac{P_t^{(n)}}{P_{t-\widetilde{\tau}}^{(n)}} \right), \tag{6.204}$$

where $n = 1, \ldots, N \equiv 8$. In our example, an estimation horizon $\widetilde{\tau}$ of one week provides a good balance in the trade-off between the number of independent observations and the homogeneity of the data.

We estimate the distribution of the invariants from currently available information. In this example, information becomes the time series of weekly compounded returns for the past five years. This is a series of approximately 250 observations. We fit the weekly compounded returns to a multivariate normal distribution:

$$\mathbf{C}_{t,\widetilde{\tau}} \sim \mathrm{N}\left(\widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}\right). \tag{6.205}$$

To estimate $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\Sigma}}$ we first compute the sample mean and the sample covariance matrix:

$$\widehat{\mathbf{m}} \equiv \frac{1}{T}\sum_{t=1}^{T}\mathbf{c}_{t,\widetilde{\tau}}, \quad \widehat{\mathbf{S}} \equiv \frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{c}_{t,\widetilde{\tau}} - \widehat{\mathbf{m}}\right)\left(\mathbf{c}_{t,\widetilde{\tau}} - \widehat{\mathbf{m}}\right)'. \tag{6.206}$$

Then as in (4.160) we shrink the covariance matrix toward a spherical estimator:

$$\widehat{\boldsymbol{\Sigma}} \equiv (1-\epsilon)\,\widehat{\mathbf{S}} + \frac{\epsilon}{N}\sum_{n=1}^{N}\widehat{S}_{nn}\mathbf{I}_N, \tag{6.207}$$

where from (4.161) the shrinkage weight reads:

$$\epsilon \equiv \frac{1}{T}\frac{\frac{1}{T}\sum_{t=1}^{T}\operatorname{tr}\left\{\left(\mathbf{c}_{t,\widetilde{\tau}}\mathbf{c}_{t,\widetilde{\tau}}' - \widehat{\mathbf{S}}\right)^2\right\}}{\operatorname{tr}\left\{\left(\widehat{\mathbf{S}} - \frac{1}{N}\sum_{n=1}^{N}\widehat{S}_{nn}\mathbf{I}_N\right)^2\right\}}. \tag{6.208}$$

Finally, as in (4.138) we shrink the sample mean towards a target vector:

$$\widehat{\boldsymbol{\mu}} \equiv (1-\gamma)\,\widehat{\mathbf{m}} + \gamma\mathbf{b}. \tag{6.209}$$

In this expression the shrinkage target follows from (4.142):

$$\mathbf{b} \equiv \frac{\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\widehat{\mathbf{m}}}{\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}\mathbf{1}; \tag{6.210}$$

and the shrinkage weight follows from (4.139) and reads in terms of the highest eigenvalue λ_1 of $\widehat{\boldsymbol{\Sigma}}$ as follows:

$$\gamma \equiv \frac{1}{T}\frac{\sum_{n=1}^{N}\widehat{\Sigma}_{nn} - 2\lambda_1}{\left(\widehat{\mathbf{m}} - \mathbf{b}\right)'\left(\widehat{\mathbf{m}} - \mathbf{b}\right)}. \tag{6.211}$$

We project the distribution of the invariants to the investment horizon by means of (3.64). As in (3.74) we obtain that compounded returns from the investment date to the investment horizon are normally distributed:

$$\mathbf{C}_{T+\tau,\tau} \sim \mathrm{N}\left(\frac{\tau}{\widetilde{\tau}}\widehat{\boldsymbol{\mu}}, \frac{\tau}{\widetilde{\tau}}\widehat{\boldsymbol{\Sigma}}\right). \tag{6.212}$$

We determine the transaction costs in the market. In this case we assume that the transaction costs grow quadratically with the number of shares transacted:

$$\mathcal{T}\left(\widetilde{\boldsymbol{\alpha}}, \boldsymbol{\alpha}\right) = \mathbf{k}'\left(\widetilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}\right) + \left(\widetilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}\right)' \mathbf{D}\left(\widetilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}\right), \tag{6.213}$$

where $\mathbf{D}$ is a diagonal matrix of positive entries. The non-linear growth of the transaction costs accounts for the market impact of large stock transactions.

6.7.3 Computing the optimal allocation

First, we formulate the investor's constraints. We assume that the investor has a budget constraint:

$$\mathcal{C}_1: \ \boldsymbol{\alpha}'\mathbf{p}_T \leq w - \mathcal{T}\left(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}\right), \tag{6.214}$$

where w is his initial capital of ten thousand dollars, and $\mathcal{T}$ are the transaction costs (6.213). Furthermore, we assume that he can only hold long positions:

$$\mathcal{C}_2: \ \boldsymbol{\alpha} \geq \mathbf{0}. \tag{6.215}$$

With the information on the investor's profile, his market and his constraints we can set up the optimization problem (6.33), which in this context reads:

$$\boldsymbol{\alpha}^* \equiv \underset{\substack{\boldsymbol{\alpha}\geq\mathbf{0} \\ \boldsymbol{\alpha}'\mathbf{p}_T \leq w - \boldsymbol{\alpha}'\mathbf{D}\boldsymbol{\alpha}}}{\operatorname{argmax}} \ \mathrm{E}\left\{\left(\boldsymbol{\alpha}' \operatorname{diag}\left(\mathbf{p}_T\right) e^{\mathbf{C}_{T+\tau,\tau}}\right)^{\frac{\gamma-1}{\gamma}}\right\}, \tag{6.216}$$

where the distribution of $\mathbf{C}$ is provided in (6.212). Since it is not possible to determine the solution $\boldsymbol{\alpha}^*$ analytically, we resort to the mean-variance framework to restrict the search to a limited number of portfolios.

We compute the inputs of the mean-variance problem (6.74), namely the expected value and the covariance of the market prices at the investment horizon. To do this, consider the characteristic function (2.157) of the compounded returns (6.205), which reads:

$$\phi_{\mathbf{C}_{t,\tilde{\tau}}}\left(\boldsymbol{\omega}\right) = e^{i\boldsymbol{\omega}'\widehat{\boldsymbol{\mu}} - \frac{1}{2}\boldsymbol{\omega}'\widehat{\boldsymbol{\Sigma}}\boldsymbol{\omega}}. \tag{6.217}$$

From (6.77) we obtain the expected value of the market prices:

$$\begin{aligned} \mathrm{E}\left\{P_{T+\tau}^{(n)}\right\} &= P_T^{(n)}\left[\phi_{\mathbf{C}_{t,\tilde{\tau}}}\left(-i\boldsymbol{\delta}^{(n)}\right)\right]^{\frac{\tau}{\tilde{\tau}}} \\ &= P_T^{(n)} e^{\frac{\tau}{\tilde{\tau}}\left(\widehat{\mu}_n + \frac{\widehat{\Sigma}_{nn}}{2}\right)}. \end{aligned} \tag{6.218}$$

Similarly, from (6.78) we obtain:

$$\begin{aligned} \mathrm{E}\left\{P_{T+\tau}^{(m)}, P_{T+\tau}^{(n)}\right\} &= P_T^{(m)} P_T^{(n)} \left[\phi_{\mathbf{C}_{t,\tilde{\tau}}}\left(-i\boldsymbol{\delta}^{(m)} - i\boldsymbol{\delta}^{(n)}\right)\right]^{\frac{\tau}{\tilde{\tau}}} \\ &= P_T^{(m)} P_T^{(n)} e^{\frac{\tau}{\tilde{\tau}}(\widehat{\mu}_m + \widehat{\mu}_n)} e^{\frac{1}{2}\frac{\tau}{\tilde{\tau}}\left(\widehat{\Sigma}_{mm} + \widehat{\Sigma}_{nn} + 2\widehat{\Sigma}_{mn}\right)}. \end{aligned} \tag{6.219}$$

Therefore from (6.79) we obtain the covariance matrix of the market:

$$\begin{aligned} \mathrm{Cov}\left\{P_{T+\tau}^{(m)}, P_{T+\tau}^{(n)}\right\} = P_T^{(m)} P_T^{(n)} e^{\frac{\tau}{\tilde{\tau}}(\widehat{\mu}_m + \widehat{\mu}_n)} \\ e^{\frac{1}{2}\frac{\tau}{\tilde{\tau}}\left(\widehat{\Sigma}_{mm} + \widehat{\Sigma}_{nn}\right)} \left(e^{\frac{\tau}{\tilde{\tau}}\widehat{\Sigma}_{mn}} - 1\right). \end{aligned} \tag{6.220}$$

With the inputs (6.218) and (6.220) we compute numerically the mean-variance efficient curve (6.74). In order to do this, we choose a significative grid of, say $I \equiv 100$ target variances $\left\{v^{(1)}, \ldots, v^{(I)}\right\}$ and solve numerically each time the following optimization:

$$\begin{aligned} \boldsymbol{\alpha}^{(i)} \equiv \underset{\boldsymbol{\alpha}}{\mathrm{argmax}}\ & \boldsymbol{\alpha}' \, \mathrm{E}\left\{\mathbf{P}_{T+\tau}\right\} \\ \text{subject to } & \begin{cases} \boldsymbol{\alpha}' \, \mathrm{Cov}\left\{\mathbf{P}_{T+\tau}\right\} \boldsymbol{\alpha} \leq v^{(i)} \\ \boldsymbol{\alpha}' \mathbf{p}_T \leq w - \boldsymbol{\alpha}' \mathbf{D} \boldsymbol{\alpha} \\ \boldsymbol{\alpha} \geq \mathbf{0}. \end{cases} \end{aligned} \tag{6.221}$$

Each optimization (6.221) is a quadratically constrained linear programming problem, i.e. a subclass of (6.57). Therefore it can be efficiently solved numerically.

In the top plot of Figure 6.23 we display the risk/reward profile of the mean-variance efficient allocations (6.221) in terms of expected value and standard deviation of final wealth:

$$\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\} = \boldsymbol{\alpha}' \, \mathrm{E}\left\{\mathbf{P}_{T+\tau}\right\}, \quad \mathrm{Sd}\left\{\Psi_{\boldsymbol{\alpha}}\right\} = \sqrt{\boldsymbol{\alpha}' \, \mathrm{Cov}\left\{\mathbf{P}_{T+\tau}\right\} \boldsymbol{\alpha}}. \tag{6.222}$$

In the middle plot of Figure 6.23 we display the mean-variance efficient allocations (6.221) in terms of their relative weights:

$$\mathbf{w}^{(i)} \equiv \frac{\mathrm{diag}\left(\boldsymbol{\alpha}^{(i)}\right) \mathbf{p}_T}{\mathbf{p}_T' \boldsymbol{\alpha}^{(i)}}. \tag{6.223}$$

According to the mean-variance optimization, the optimal allocation is the portfolio (6.221) that gives rise to the higher level of satisfaction. To determine this portfolio we use Monte Carlo simulations.

We simulate a large number J of Monte Carlo market scenarios as follows:

$${}_j\mathbf{P}_{T+\tau} \equiv \mathrm{diag}\left(\mathbf{p}_T\right) e^{{}_j\mathbf{C}}, \tag{6.224}$$

where the exponential acts component-wise and where each vector ${}_j\mathbf{C}$ is an independent drawing from the multivariate normal distribution (6.212) for all $j = 1, \ldots, J$. In our example we perform $J \equiv 10^5$ simulations.

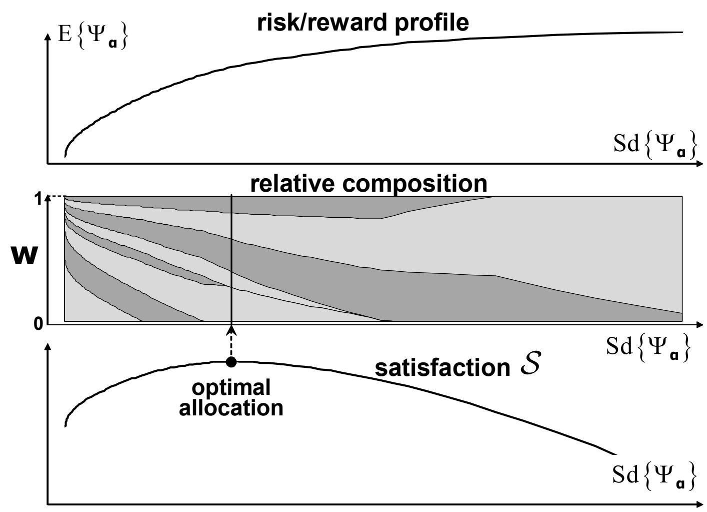

Fig. 6.23. MV approach: two-step allocation optimization

We evaluate numerically the satisfaction drawn from each of the mean-variance efficient allocations. In other words, we compute the following approximation to (6.203) for all the mean-variance efficient portfolios (6.221) in the grid:

$$\widetilde{\mathcal{S}}\left(\boldsymbol{\alpha}^{(i)}\right) \equiv \left(\frac{\gamma}{J}\sum_{j=1}^{J}\frac{\left({}_{j}\mathbf{P}_{T+\tau}'\boldsymbol{\alpha}^{(i)}\right)^{\gamma}}{\gamma}\right)^{\frac{1}{\gamma}}. \tag{6.225}$$

In the bottom plot in Figure 6.23 we display the satisfaction (6.225) ensuing from each of the allocations in the grid.

We rank the levels of satisfaction provided by the mean-variance efficient portfolios:

$$i^{*} \equiv \underset{i}{\operatorname{argmax}}\left\{\widetilde{\mathcal{S}}\left(\boldsymbol{\alpha}^{(i)}\right)\right\}. \tag{6.226}$$

Finally, we determine the optimal allocation:

$$\boldsymbol{\alpha}^{*} \equiv \boldsymbol{\alpha}^{(i^{*})}, \tag{6.227}$$

see Figure 6.23.

$$\mathbf{z}^{*} \equiv \underset{\mathbf{z}}{\operatorname{argmin}} \left\{ \left\| \mathbf{A}_{(0)}^{1/2} \mathbf{E}_{(0)}' \mathbf{z} + \mathbf{A}_{(0)}^{-1/2} \mathbf{E}_{(0)}' \mathbf{u}_{(0)} \right\|^{2} \right\} \tag{T6.15}$$

$$\text{s.t.} \begin{cases} \mathbf{A}\mathbf{z} = \mathbf{a} \\ \left\| \mathbf{A}_{(j)}^{1/2} \mathbf{E}_{(j)}' \mathbf{z} + \mathbf{A}_{(j)}^{-1/2} \mathbf{E}_{(j)}' \mathbf{u}_{(j)} \right\|^{2} \leq \mathbf{u}_{(j)} \mathbf{S}_{(j)}^{-1} \mathbf{u}_{(j)} - v_{(j)}, \end{cases}$$

~~for $j = 1, \ldots, J$. Introducing a new variable t this problem is equivalent to:~~

$$(\mathbf{z}^{*}, t^{*}) \equiv \underset{(\mathbf{z}, t)}{\operatorname{argmin}} \{t\} \tag{T6.16}$$

$$\text{s.t.} \begin{cases} \mathbf{A}\mathbf{z} = \mathbf{a} \\ \left\| \mathbf{A}_{(0)}^{1/2} \mathbf{E}_{(0)}' \mathbf{z} + \mathbf{A}_{(0)}^{-1/2} \mathbf{E}_{(0)}' \mathbf{u}_{(0)} \right\| \leq t \\ \left\| \mathbf{A}_{(1)}^{1/2} \mathbf{E}_{(1)}' \mathbf{z} + \mathbf{A}_{(1)}^{-1/2} \mathbf{E}_{(1)}' \mathbf{u}_{(1)} \right\| \leq \sqrt{\mathbf{u}_{(1)} \mathbf{S}_{(1)}^{-1} \mathbf{u}_{(1)} - v_{(1)}} \\ \vdots \\ \left\| \mathbf{A}_{(J)}^{1/2} \mathbf{E}_{(J)}' \mathbf{z} + \mathbf{A}_{(J)}^{-1/2} \mathbf{E}_{(J)}' \mathbf{u}_{(J)} \right\| \leq \sqrt{\mathbf{u}_{(J)} \mathbf{S}_{(J)}^{-1} \mathbf{u}_{(J)} - v_{(J)}} \end{cases}$$

6.3 Feasible set and MV efficient frontier

To solve

$$\boldsymbol{\alpha}(v) \equiv \underset{\boldsymbol{\alpha}'\mathbf{d} = c, \operatorname{Var}\{\Psi_{\boldsymbol{\alpha}}\} = v}{\operatorname{argmax}} \operatorname{E}\{\Psi_{\boldsymbol{\alpha}}\}, \tag{T6.17}$$

we first compute the feasible set in the space of moments of the objective function $(v, e) = (\operatorname{Var}\{\Psi_{\boldsymbol{\alpha}}\}, \operatorname{E}\{\Psi_{\boldsymbol{\alpha}}\})$.

We consider the general case where $\operatorname{E}\{\mathbf{M}\}$ and $\mathbf{d}$ are not collinear. First we prove that any level of expected value $e \in \mathbb{R}$ is attainable. This is true if for any value e there exists an $\boldsymbol{\alpha}$ such that:

$$e = \operatorname{E}\{\Psi_{\boldsymbol{\alpha}}\} = \boldsymbol{\alpha}' \operatorname{E}\{\mathbf{M}\} \tag{T6.18}$$

$$c = \boldsymbol{\alpha}'\mathbf{d}. \tag{T6.19}$$

In turn, this is true if we can solve the following system for an arbitrary value of e:

$$\begin{pmatrix} \operatorname{E}\{M_j\} & \operatorname{E}\{M_k\} \\ b_j & b_k \end{pmatrix} \begin{pmatrix} \alpha_j \\ \alpha_k \end{pmatrix} = \begin{pmatrix} e - \sum_{n \neq j,k} \alpha_n \operatorname{E}\{M_k\} \\ c - \sum_{n \neq j,k} \alpha_n b_n \end{pmatrix}. \tag{T6.20}$$

Since $\operatorname{E}\{\mathbf{M}\}$ and $\mathbf{d}$ are not collinear we can always find two indices (j, k) such that the matrix on the left-hand side of $(T6.20)$ is invertible. Therefore, we can fix arbitrarily e and all the entries of $\boldsymbol{\alpha}$ that appear on the right hand side of $(T6.20)$ and solve for the remaining two entries on the left-hand side of $(T6.20)$.

Now we prove that if a point (v, e) is feasible, so is any point $(v + \gamma, e)$, where γ is any positive number. Indeed, if we make any of the entries on the right hand side of (T6.20) go to infinity and solve for the remaining two entries on the left-hand side of (T6.20) the variance of the ensuing allocations satisfies the constraints and tends to infinity. For continuity, all the points between (v, e) and $(+\infty, e)$ are covered.

Therefore the feasible set can only be bounded on the left of the (v, e) plane. To find out if that boundary exists, we fix a generic expected value e and compute the minimum variance achievable that satisfies the affine constraint. Therefore, we minimize the following unconstrained Lagrangian:

$$\begin{aligned}\mathcal{L}(\boldsymbol{\alpha}, \lambda, \mu) &\equiv \mathrm{Var}\{\Psi_{\boldsymbol{\alpha}}\} - \lambda(\boldsymbol{\alpha}'\mathbf{d} - c) - \mu(\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} - e). \\ &= \boldsymbol{\alpha}' \mathrm{Cov}\{\mathbf{M}\}\boldsymbol{\alpha} - \lambda(\boldsymbol{\alpha}'\mathbf{d} - c) - \mu(\boldsymbol{\alpha}' \mathrm{E}\{\mathbf{M}\} - e).\end{aligned} \tag{T6.21}$$

The first-order conditions yield:

$$\mathbf{0} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{\alpha}} = 2\,\mathrm{Cov}\{\mathbf{M}\}\boldsymbol{\alpha} - \lambda\mathbf{d} - \mu\,\mathrm{E}\{\mathbf{M}\} \tag{T6.22}$$

in addition to the two constraints

$$\begin{aligned}0 &= \frac{\partial \mathcal{L}}{\partial \lambda} = \boldsymbol{\alpha}'\mathbf{d} - c \\ 0 &= \frac{\partial \mathcal{L}}{\partial \mu} = \boldsymbol{\alpha}' \mathrm{E}\{\mathbf{M}\} - e,\end{aligned} \tag{T6.23}$$

From (T6.22) the solution reads

$$\boldsymbol{\alpha} = \frac{\lambda}{2}\mathrm{Cov}\{\mathbf{M}\}^{-1}\mathbf{d} + \frac{\mu}{2}\mathrm{Cov}\{\mathbf{M}\}^{-1}\mathrm{E}\{\mathbf{M}\}. \tag{T6.24}$$

The Lagrange multipliers can be obtained as follows: First, we define four scalar constants:

$$\begin{array}{ll} A \equiv \mathbf{d}' \mathrm{Cov}\{\mathbf{M}\}^{-1}\mathbf{d} & B \equiv \mathbf{d}'\mathrm{Cov}\{\mathbf{M}\}^{-1}\mathrm{E}\{\mathbf{M}\} \\ C \equiv \mathrm{E}\{\mathbf{M}\}'\mathrm{Cov}\{\mathbf{M}\}^{-1}\mathrm{E}\{\mathbf{M}\} & D \equiv AC - B^2 \end{array} \tag{T6.25}$$

Left-multiplying the solution (T6.24) by $\mathbf{d}'$ and using the first constraint in (T6.23) we obtain:

$$\begin{aligned}c = \mathbf{d}'\boldsymbol{\alpha} &= \frac{\lambda}{2}\mathbf{d}'\mathrm{Cov}\{\mathbf{M}\}^{-1}\mathbf{d} \\ &\quad + \frac{\mu}{2}\mathbf{d}'\mathrm{Cov}\{\mathbf{M}\}^{-1}\mathrm{E}\{\mathbf{M}\} \\ &= \frac{\lambda}{2}A + \frac{\mu}{2}B.\end{aligned} \tag{T6.26}$$

Similarly, left-multiplying the solution (T6.24) by $\mathrm{E}\{\mathbf{M}\}'$ and using the second constraint in (T6.23) we obtain:

$$
\begin{aligned}
e = \mathrm{E}\{\mathbf{M}\}' \boldsymbol{\alpha} &= \frac{\lambda}{2} \mathrm{E}\{\mathbf{M}\}' \operatorname{Cov}\{\mathbf{M}\}^{-1} \mathbf{d} \\
&+ \frac{\mu}{2} \mathrm{E}\{\mathbf{M}\}' \operatorname{Cov}\{\mathbf{M}\}^{-1} \mathrm{E}\{\mathbf{M}\} \\
&= \frac{\lambda}{2} B + \frac{\mu}{2} C
\end{aligned} \tag{T6.27}
$$

Now we can invert ($T6.27$) and ($T6.26$) obtaining:

$$
\lambda = \frac{2cC - 2eB}{D}, \qquad \mu = \frac{2eA - 2cB}{D} \tag{T6.28}
$$

Finally, left-multiplying ($T6.22$) by $\boldsymbol{\alpha}'$ we obtain:

$$
\begin{aligned}
0 &= 2\boldsymbol{\alpha}' \operatorname{Cov}\{\mathbf{M}\} \boldsymbol{\alpha} - \lambda \boldsymbol{\alpha}' \mathbf{d} - \mu \boldsymbol{\alpha}' \mathrm{E}\{\mathbf{M}\} \\
&= 2 \operatorname{Var}\{\Psi_{\boldsymbol{\alpha}}\} - \lambda c - \mu e \\
&= 2 \left(\operatorname{Var}\{\Psi_{\boldsymbol{\alpha}}\} - \frac{cC - eB}{D} c - \frac{eA - cB}{D} e \right).
\end{aligned} \tag{T6.29}
$$

This shows that the boundary $v(e) \equiv \operatorname{Var}\{\Psi_{\boldsymbol{\alpha}}\}$ exists. Collecting the terms in e we obtain its equation:

$$
v = \frac{A}{D} e^2 - \frac{2cB}{D} e + \frac{c^2 C}{D}, \tag{T6.30}
$$

which shows that the feasible set is bounded on the left by a parabola. In the space of the coordinates $(d, e) = (\operatorname{Sd}\{\Psi_{\boldsymbol{\alpha}}\}, \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\})$ the parabola ($T6.30$) becomes a hyperbola:

$$
d^2 = \frac{A}{D} e^2 - \frac{2cB}{D} e + \frac{c^2 C}{D}, \tag{T6.31}
$$

The allocations $\boldsymbol{\alpha}$ that give rise to the boundary parabola ($T6.30$) are obtained from ($T6.24$) by substituting the Lagrange multipliers ($T6.28$):

$$
\begin{aligned}
\boldsymbol{\alpha} &= \frac{cC - eB}{D} \operatorname{Cov}\{\mathbf{M}\}^{-1} \mathbf{d} + \frac{eA - cB}{D} \operatorname{Cov}\{\mathbf{M}\}^{-1} \mathrm{E}\{\mathbf{M}\} \\
&= \frac{(cC - eB) A}{D} \frac{\operatorname{Cov}\{\mathbf{M}\}^{-1} \mathbf{d}}{\mathbf{d}' \operatorname{Cov}\{\mathbf{M}\}^{-1} \mathbf{d}} \\
&+ \frac{(eA - cB) B}{D} \frac{\operatorname{Cov}\{\mathbf{M}\}^{-1} \mathrm{E}\{\mathbf{M}\}}{\mathbf{d}' \operatorname{Cov}\{\mathbf{M}\}^{-1} \mathrm{E}\{\mathbf{M}\}}
\end{aligned} \tag{T6.32}
$$

If $c \neq 0$ we can write ($T6.32$) as:

$$
\boldsymbol{\alpha} = (1 - \gamma(\boldsymbol{\alpha})) \boldsymbol{\alpha}_{MV} + \gamma(\boldsymbol{\alpha}) \boldsymbol{\alpha}_{SR}, \tag{T6.33}
$$

where the scalar γ is defined as:

$$\gamma(\boldsymbol{\alpha}) \equiv \frac{(\mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} A - cB) B}{cD} \qquad (T6.34)$$

and $(\boldsymbol{\alpha}_{MV}, \boldsymbol{\alpha}_{SR})$ are two specific portfolios defined as follows:

$$\boldsymbol{\alpha}_{MV} \equiv \frac{c \operatorname{Cov}\{\mathbf{M}\}^{-1} \mathbf{d}}{\mathbf{d}' \operatorname{Cov}\{\mathbf{M}\}^{-1} \mathbf{d}} \qquad (T6.35)$$

$$\boldsymbol{\alpha}_{SR} \equiv \frac{c \operatorname{Cov}\{\mathbf{M}\}^{-1} \mathrm{E}\{\mathbf{M}\}}{\mathbf{d}' \operatorname{Cov}\{\mathbf{M}\}^{-1} \mathrm{E}\{\mathbf{M}\}}. \qquad (T6.36)$$

Portfolio ($T6.35$) corresponds to the case $\gamma = 0$. From the expression for γ in ($T6.34$) and from the expression for the Lagrange multipliers in ($T6.28$) we see that $\boldsymbol{\alpha}_{MV}$ is the allocation that corresponds to the case where the Lagrange multiplier μ is zero in ($T6.24$). From the original Lagrangian ($T6.21$), if $\mu = 0$ the ensuing allocation is the minimum-variance portfolio. From ($T6.30$), or by direct computation we derive the coordinates of $\boldsymbol{\alpha}_{MV}$ in the space of moments:

$$v_{MV} \equiv \operatorname{Var}\{\Psi_{\boldsymbol{\alpha}_{MV}}\} = \frac{c^2}{A}, \quad e_{MV} \equiv \mathrm{E}\{\Psi_{\boldsymbol{\alpha}_{MV}}\} = \frac{cB}{A}. \qquad (T6.37)$$

Portfolio ($T6.36$) corresponds to the case $\gamma = 1$. This is the allocation on the feasible boundary that corresponds to the highest Sharpe ratio. Indeed, by direct computation we derive the coordinates of $\boldsymbol{\alpha}_{SR}$ in the space of moments:

$$v_{SR} \equiv \operatorname{Var}\{\Psi_{\boldsymbol{\alpha}_{SR}}\} = \frac{c^2 C}{B^2}, \quad e_{SR} \equiv \mathrm{E}\{\Psi_{\boldsymbol{\alpha}_{SR}}\} = \frac{cC}{B}, \qquad (T6.38)$$

On the other hand the highest Sharpe ratio is the steepness of the straight line tangent to the hyperbola ($T6.31$), which we obtain by maximizing its analytical expression as a function of the expected value:

$$\mathrm{SR}(e) \equiv \frac{e}{d(e)} = \frac{e}{\sqrt{\frac{A}{D}e^2 - \frac{2cB}{D}e + \frac{c^2 C}{D}}}. \qquad (T6.39)$$

The first-order conditions with respect to e show that the maximum of the Sharpe ratio is reached at ($T6.38$).

It is immediate to check that the ratio e/v is the same for both portfolio ($T6.37$) and portfolio ($T6.38$), and thus the two allocations lie on the same radius from the origin in the (v, e) plane.

As for the expression of the scalar γ in ($T6.34$), since

$$\mathrm{E}\{\Psi_{\boldsymbol{\alpha}_{SR}}\} - \mathrm{E}\{\Psi_{\boldsymbol{\alpha}_{MV}}\} = \frac{cC}{B} - \frac{cB}{A} = \frac{cD}{AB} \qquad (T6.40)$$

we can simplify it as follows:

$$\begin{aligned}\gamma &\equiv \frac{\left(\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\} A - cB\right) B}{cD} = \frac{\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\} AB}{cD} - \frac{B^2}{D} \\ &= \frac{\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\}}{\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}_{SR}}\right\} - \mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}_{MV}}\right\}} - \frac{\left(\frac{cB}{A}\right)}{\left(\frac{cD}{AB}\right)} \\ &= \frac{\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\} - \mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}_{MV}}\right\}}{\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}_{SR}}\right\} - \mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}_{MV}}\right\}},\end{aligned} \tag{T6.41}$$

which shows that the upper (lower) branch of the boundary parabola is spanned by the positive (negative) values of γ.

To consider the case $c = 0$ we take the limit $c \to 0$ in the above results. The boundary (*T*6.30) of the feasible set in the coordinates $(v, e) = (\mathrm{Var}\left\{\Psi_{\boldsymbol{\alpha}}\right\}, \mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\})$ is still a parabola:

$$v = \frac{A}{D} e^2; \tag{T6.42}$$

whereas in the space of coordinates $(s, e) = (\mathrm{Sd}\left\{\Psi_{\boldsymbol{\alpha}}\right\}, \mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\})$ the boundary degenerates from the hyperbola (*T*6.31) into two straight lines:

$$d\left(e\right) = \pm\sqrt{\frac{A}{D}} e. \tag{T6.43}$$

As for the allocations that generate this boundary, taking the limit $c \to 0$ in (*T*6.33) and recalling the definitions (*T*6.34), (*T*6.35) and (*T*6.36) we obtain:

$$\begin{aligned}\boldsymbol{\alpha} &= \lim_{c\to 0}\left[\boldsymbol{\alpha}_{MV} + \gamma\left(\boldsymbol{\alpha}\right)\left(\boldsymbol{\alpha}_{SR} - \boldsymbol{\alpha}_{MV}\right)\right] \\ &= \lim_{c\to 0}\left[\gamma\left(\boldsymbol{\alpha}\right)\left(\boldsymbol{\alpha}_{SR} - \boldsymbol{\alpha}_{MV}\right)\right] \\ &= \mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\}\frac{\mathrm{Cov}\left\{\mathbf{M}\right\}^{-1}}{D}\left(A\,\mathrm{E}\left\{\mathbf{M}\right\} - B\mathbf{d}\right) \\ &= \zeta\left(\boldsymbol{\alpha}\right)\mathrm{Cov}\left\{\mathbf{M}\right\}^{-1}\left(A\,\mathrm{E}\left\{\mathbf{M}\right\} - B\mathbf{d}\right),\end{aligned} \tag{T6.44}$$

where the scalar ζ is defined as follows

$$\zeta\left(\boldsymbol{\alpha}\right) \equiv \frac{\mathrm{E}\left\{\Psi_{\boldsymbol{\alpha}}\right\}}{D}. \tag{T6.45}$$

The upper (lower) branch of the boundary parabola is spanned by the positive (negative) values of ζ.

With the geometry of the feasible set at hand, we can move on to compute the mean-variance curve (*T*6.17): fixing a level of variance v and maximizing the expected value in the feasible set means hitting the upper branch of the parabola (*T*6.30). Therefore if $c \neq 0$ the mean-variance curve reads:

$$\boldsymbol{\alpha} \equiv \left(1 - \gamma\right)\boldsymbol{\alpha}_{MV} + \gamma\boldsymbol{\alpha}_{SR}, \qquad \gamma > 0. \tag{T6.46}$$

if $c = 0$ the mean-variance curve reads:

$$\boldsymbol{\alpha} \equiv \zeta\frac{\mathrm{Cov}\left\{\mathbf{M}\right\}^{-1}\left(\mathrm{E}\left\{\mathbf{M}\right\} - \mathbf{d}\right)}{\mathbf{d}'\,\mathrm{Cov}\left\{\mathbf{M}\right\}^{-1}\mathrm{E}\left\{\mathbf{M}\right\}}, \qquad \zeta\,\mathrm{sign}\left(B\right) > 0. \tag{T6.47}$$